INTRODUCTION TO
THEORETICAL PHYSICS

INTRODUCTION TO THEORETICAL PHYSICS

VOLUME I

BY

ARTHUR HAAS, Ph.D.

PROFESSOR OF PHYSICS IN THE UNIVERSITY OF VIENNA

TRANSLATED FROM THE THIRD AND FOURTH
EDITIONS BY

T. VERSCHOYLE, M.C., Ph.D., A.R.C.S.

WITH A FOREWORD BY

PROF. F. G. DONNAN, F.R.S.

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FOREWORD

A SURE understanding of the quantitative principles of physical science is nowadays a necessity for everybody—for the chemist, engineer, and physiologist as much as for the physicist. Nobody can afford to say that he is not a physicist and so expect to be forgiven for his ignorance. Nevertheless, the difficulty is a real enough one for the many who fear the mathematical reasoning and the wealth and subtlety of physical science. It is scarcely going too far to say that Heaven must have sent Professor Haas in answer to many prayers, for in the present work he has solved the formidable problem of teaching us the whole of the principles of mathematical physics by a wonderful combination of rigorous science and rare art. The mathematical difficulties are explained step by step—there are no gaps in the reasoning left for expert readers to bridge and non-expert ones to weep over. The whole structure of physical theory is built up piece by piece, with an exposition so sequent and crystal clear that we can pass through and understand the great luminous building without painful effort. We see first traced before our eyes the great conquests of " classical " physics, and then the paths by which these lead up to and merge into the splendid advances—the finer analysis and the profounder synthesis—of modern times.

In this arrangement and division of the subject-matter, the author has shown himself to be not only a wise and experienced teacher, but also a true philosopher. Professor Haas has done an immense service for all students of science, and the appearance of his book is an event of first-class importance. Certainly every student of physics will read it without any recommendation from me. I cannot too strongly urge every student of chemistry, and every student of biochemistry and biophysics, to do the same. There was

a time when the chemist could exist with "heat, light, sound, and a little electricity," garnished with nothing more piquant than quadratic equations. Indeed, anything stronger was supposed to give the poor fellow mental indigestion. Nowadays the chemist requires all the help that the physicist can give him, if he is to succeed in his many quests. He will never find a better friend than Professor Haas, so I heartily welcome the appearance of this English translation of his book made by Mr. Verschoyle.

<div style="text-align: right">F. G. DONNAN.</div>

UNIVERSITY COLLEGE,
     LONDON,
         April, 1924.

PREFACE

TO THE ENGLISH EDITION

THE complete transformation which, in recent years, our views on natural phenomena have undergone, as a result of the wonderful advances made in theoretical physics, has awakened in wide circles a lively interest in the problems of this science. There are many who lack the time or preliminary knowledge necessary for the study of the original treatises and works on modern theoretical physics, which are devoted to special individual subjects. Such students may, perhaps, cherish a wish for an exposition which, while modern in treatment and outlook and not too comprehensive, would give a survey of the present state of theoretical physics as a whole, so as to lead to an exact understanding of the fundamental principles and the chief problems of the science, without going too far into details.

To provide such an exposition, in a form also suitable for university courses, is the object of the present book. It has been compiled from lectures which I delivered at the Universities of Leipzig and Vienna, and in which my object was, as far as possible, to combine " classical " and modern physics into one united whole.

As regards mathematics, only the principal rules of differential and integral calculus are assumed as known in this book. In calculations even easy intermediate steps are purposely always carried out in full. When the reader is but a beginner—as must often be the case in the reading of textbooks—and has to rack his brains for hours over some intermediate calculation called simple, but not given, his attention will all too likely be deflected from important physical questions to trivial mathematical points. With the object of avoiding absolutely this possibility in the study of this book, mathematical considerations have been given in too great, rather than in too little, detail.

I have further endeavoured to exclude from the physical considerations all processes of thought essentially purely mathematical in character, and to collect them in separate sections in a quite general and abstract form, without any

reference to physical phenomena. In this way I hope that not only will the real physical processes of thought, uninterrupted by any mathematical considerations, gain in clarity, but also that it may be more evident to the reader which relations between physical theorems are purely mathematical in nature, and which can only be established with the assistance of experimental facts of physics. The development of a uniform vectorial method, to be applied both in mechanics and in the theories of electricity and relativity, permits of a simplification of many derivations, and saves superfluous repetitions of analogous processes of thought in different branches of physics.

The division of the subject-matter was determined by the desire for a sharp separation between those physical relations that can be derived independently of all atomistic hypotheses and true atomistic physics. In the first volume only relations of the first kind are treated, although, in the theory of the electromagnetic field, fundamental assumptions of atomic physics have been introduced, provided that they can actually be derived in a more general form without the aid of any atomistic hypotheses (*e.g.*, convection currents, the equation of motion of an electric charge, electromagnetic mass).

In this book I have departed from the usual treatment, principally in the case of thermodynamics. It seemed to me, indeed, that at the present time the statistical meaning of the phenomena of heat is no longer just one of several possible interpretations, but is the one demanded by the fact of the actual existence of the atoms. Conversely, it becomes necessary to find a physical interpretation of the laws derived by general statistical methods for the individual partition of energy. Thermodynamics are accordingly based in this book on pure statistics, in as general a form as possible and without requiring any special assumptions as to the nature of the individually distributed energy, thus leading to the two principal laws in their generally valid form. The special application, however, on the one hand, to translatory energy of motion, and, on the other hand, to oscillatory energy quanta, both of a mechanical and of an electromagnetic kind, gives the laws of gases, of extremely cold bodies, and of thermal radiation. Although the statistical derivation of the principal laws of thermodynamics is still by no means complete, it appeared to me that the decision must be in favour of the statistical rather than the phenomenological method. For only the statistical method can com-

bine the theory of heat with the unitary system of modern physics ; only the statistical method can make the otherwise unintelligible irreversibility intelligible ; finally, only the statistical method can answer the demand that theoretical physics, in its progress forward, shall completely free itself from all humanly subjective points of view.

I have intentionally left historical considerations in the background and, neglecting the often fortuitous historical order of progress, I have throughout chosen that method of exposition which, in the present state of physics and in connection with its modern problems, seemed to be the simplest and didactically the most practical.

<div align="right">ARTHUR HAAS.</div>

TRANSLATOR'S NOTE TO THE FIRST EDITION.

This translation has been made in accordance with the third and fourth German editions. The German notation has been preserved throughout, except in the representation of vector quantities.

The translator owes his very sincere thanks to Dr. R. W. Lawson, of the University of Sheffield, for his most careful revision of the manuscript and for his numerous valuable suggestions. He is also greatly indebted to Professor Haas for his very helpful advice and criticism, and to Professor Donnan for so kindly acting as " sponsor " to the work.

<div align="right">T. VERSCHOYLE.</div>

TRANSLATOR'S NOTE TO THE SECOND ENGLISH EDITION.

The alterations of consequence made in the present edition comprise the addition of part of a section on the " Hamiltonian function " and the " canonical equations of motion," and the complete rewriting of the section on " stress." All such new matter has very kindly been supplied by Professor Haas himself.

<div align="right">T. VERSCHOYLE.</div>

CONTENTS

PART I

MECHANICS

Together with the General Theory of Vector Fields, of Vibrations, and of Potential

CHAPTER I

THE MOTION OF A FREE MATERIAL PARTICLE

CHAPTER II

THE GENERAL PRINCIPLES OF MECHANICS

CHAPTER III

THE MOTION OF RIGID BODIES

CHAPTER IV

GENERAL THEORY OF VECTOR FIELDS

CHAPTER V

GENERAL THEORY OF VIBRATIONS AND OF WAVES

CHAPTER VI

THE MOTION OF DEFORMABLE BODIES

CHAPTER VII

THEORY OF POTENTIAL

PART II

THEORY OF THE ELECTROMAGNETIC FIELD AND OF LIGHT

CHAPTER VIII

ELECTRICITY AND MAGNETISM

CHAPTER IX

MAXWELL'S THEORY OF THE ELECTROMAGNETIC FIELD

CHAPTER X

THEORY OF LIGHT

APPENDIX

NOTE.—*The reader who finds §§ 20, 21, 22 and 83, 84, 85 too difficult may omit these sections, particularly as a knowledge of them is not necessary for later considerations.*

INTRODUCTION TO THEORETICAL PHYSICS

PART I

MECHANICS

Together with the General Theory of Vector Fields, of Vibrations,
and of Potential

THE MOTION OF A FREE MATERIAL PARTICLE

§ 1. The Principle of Inertia, and the Conception of Force.

THE simplest form of motion is *rectilinear motion* in which *equal distances* are traversed in *equal times*; such motion is termed *uniform*. The *velocity* of motion is defined as the ratio of any given length of the path to the time required in traversing it, and, in the case of uniform motion, its value remains constant and independent of the length of path considered.

If, now, a body describes an entirely *arbitrary* path, which will in general be *curved*, and if its motion is likewise arbitrary, then for a *short* length of path, the motion can also be regarded as being *approximately uniform*. The shorter the length we consider, the closer the approximation of the imaginary uniform and rectilinear motion to the actual non-uniform and curved motion between two neighbouring points along the path. Hence if ds is the element of length along the path which is described in the element of time dt, we may define the *differential coefficient ds/dt* as the *instantaneous value of the velocity*. We may also ascribe an *instantaneous direction* to the velocity, namely, the direction of the element of length, or in other words, that of the *tangent to the path*.

The first fundamental principle of mechanics is the *principle of inertia*, originally propounded by *Descartes* (1644) and later formulated by *Newton* (1687) as the *First Law of Motion*.[1] According to this law, every body maintains its

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[1] The principle of inertia is really due to *Galileo*. He originated the conception of ideal motion free from all obstacles, and created for this ideal motion a supreme axiom in the principle of the complete *reversibility* of the ideal mechanical process. From this principle Galileo quite correctly concluded that every motion which takes place under the influence of gravity, with a certain acceleration in the downward direction along an inclined plane, must also take place in the reverse direction, *i.e.*, upwards, with an equally large retardation (assuming, of course, the absence of all obstacles to the motion). He was further able to deduce that, in the case of an ideal *horizontal* motion, neither acceleration nor retardation is possible, and hence that such a motion, where the body is not subjected to the influence of gravity, must *continue* for ever with constant velocity. Descartes was the first to grasp the full meaning of this

velocity and its *direction of motion* unaltered, provided that *no external causes* affect its motion; and, in particular, a body that is at rest *remains at rest*, in the absence of such causes.

Consequently, if the velocity or direction of motion of a body changes, the reason for this change is to be sought in some external cause which would set the body in motion were it at rest and free to move, and which, by an old-established expression, is designated *force*.

Now it is an important fact of *experience* that we can prevent any force from setting a perfectly free body in motion by applying a *pull* with a *weight* of definite size in a definite direction.[2] Hence we can determine a force by stating how large the weight must be, and in what direction the pull must be exercised, in order to *annul* the effect of the force in question. By comparing this weight with the unit of weight, we can measure the *magnitude of the force* ; and, since we know by experience that two *equally large but opposite* pulls annul each other, we must ascribe to the force a direction exactly opposite to that of the compensating pull. As was first pointed out by *Stevin* (about 1600), every force can therefore be symbolically represented by a *line* which has the same *direction* as the force, and which contains as many units of *length* as the force contains units of force.

§ 2. Vector Quantities.

In *velocities* and *forces* we have become acquainted with quantities which are only completely determined when their *direction* is given, and which can hence be symbolically represented by *directed intervals*. Such quantities are termed *vector quantities*, or, like the directed intervals themselves, simply *vectors*.

Three essential properties are to be distinguished in every vector : the *magnitude*, the *direction*, and the *sense*. By the magnitude of a vector is to be understood the number expressing how many units the quantity in question con-

incidental theorem of Galileo's, at the same time supplementing it by the statement that a body always strives to continue its motion *only in a straight line*.   Newton gave permanent form to this principle of inertia in his first Law of Motion (*Lex Motus* I.) : " Corpus omne perseverare in statu suo quiescendi vel movendi uniformiter in directum, nisi quatenus a viribus impressis cogitur statum illum mutare " (" Every body continues in its state of rest or of uniform motion in a straight line, except in so far as it is compelled by impressed forces to change that state ").

[2] As an example, take the case of a free magnet pole attracted by a fixed one.

tains ; by the magnitude of a force, or of a velocity, for example, we mean the number of units of force, or of velocity, contained respectively in the force, or velocity. In representing a vector by a directed interval, the *length* of the latter naturally gives the magnitude of the vector.

As is customary, vectors will be distinguished in this book by *Clarendon* type. The *magnitude* of a vector will be denoted by the corresponding letter in *italic* type, *e.g.*, the magnitude of a vector **A** by A.

The agreement between two vectors **A** and **B** in magnitude, direction and sense will be symbolically expressed by the equation

(1) $$\mathbf{A} = \mathbf{B}.$$

Thus, two vectors are to be regarded as *identical* when *drawn from different points*, provided that they agree in magnitude, direction and sense. The fact that two vectors **A** and **C**, while agreeing in magnitude and direction, are opposite in sense is expressed by the symbolical equation

(2) $$\mathbf{A} = -\mathbf{C}.$$

The *projections* of a vector on the *axes of a coordinate system* are termed the *components of the vector* relative to this system. If A_x, A_y, A_z are the components of a vector **A**,[1] it follows from the theorem of Pythagoras that

(3) $$A^2 = A_x{}^2 + A_y{}^2 + A_z{}^2.$$

The square of the magnitude of a vector is thus equal to the sum of the squares of its components.

Not only the magnitude of a vector, however, is determined by the components, but also its direction ; and the sign of the components naturally gives its sense. The projection on the x-axis is, in fact, equal to the magnitude of the vector multiplied by the cosine of the angle included between the direction of the vector and the x-axis, *i.e.*,

(4) $$A_x = A \cos (\mathbf{A}, x),$$

or by eqn. (3),

(5) $$\cos (\mathbf{A}, x) = \frac{A_x}{\sqrt{A_x{}^2 + A_y{}^2 + A_z{}^2}}.$$

Two analogous equations hold for the angles formed by the direction of the vector with the y- and z-axes.

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[1] In this book the components of a vector will always be denoted by *italic* letters.

We must here make a short digression on the subject of *space coordinate systems.* There are two possible varieties of space coordinate systems, which can never be made to coincide, because one is the *mirror-image* of the other. For, if we draw the *x*- and *z*-axes in a vertical plane, the positive *y*-axis can be directed either to the rear or to the front. In the first case (Fig. 1), looking from a point on the positive *z*-axis, we see that the rotation which would carry the positive *x*-axis round to the direction of the positive *y*-axis, by the shortest way,[2] is *anti-clockwise*; in the second case (Fig. 2), this rotation appears clockwise. In the first instance we speak of an *English* coordinate
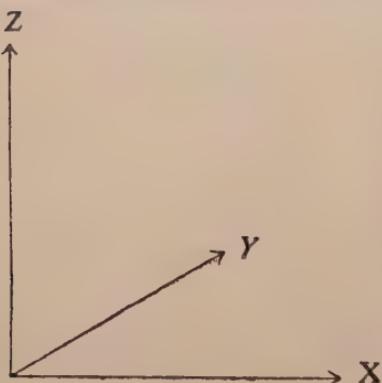
Fig. 1.
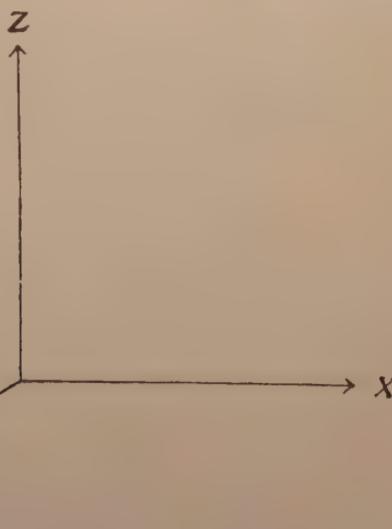
Fig. 2.

system, and in the second of a *French*[3]; the former is also called a *right-handed* system, and the latter a *left-handed*.

[2] We can, of course, turn the positive *x*-axis into the direction of the positive *y*-axis by an opposite rotation through 270°; we therefore add the condition "by the shortest way."

[3] The explanation of these terms lies in the earlier use of the system represented in Fig. 1 chiefly by English physicists, and of that represented in Fig. 2 by French physicists.

For, if we indicate the x-axis with the thumb, the y-axis with the forefinger, and the z-axis with the middle finger, we can represent the English system with the right hand, and the French system with the left.

Since the *English* system is preferable for the investigation of electromagnetic processes, it is the one now generally used in theoretical physics, and so we shall use it exclusively in what follows.

In contrast to vectors, quantities which are completely determined by a number alone, there being no question of direction, are called *scalars*. They are so termed because they are completely defined as soon as their magnitude, as measured on a definite scale, is known. As instances of scalars we may mention temperature, mass, electric charge, quantity of magnetism.

The *multiplication of a vector by a scalar* obviously consists in multiplying the magnitude of the vector by the scalar without altering the direction of the former. Similarly, the division of a vector by a scalar naturally consists in dividing its magnitude and leaving its direction unaltered.

This rule leads us further to the important conception of a *unit-vector*, by which we understand a vector whose length is equal to the *unit of length*. Every possible direction can thus be determined by a unit-vector, and so we can regard any given vector as the product of a unit-vector in the direction of the vector and of a scalar equal to the magnitude of the vector. Thus, if we denote by **a** the unit-vector falling in the direction of the vector **A**, we have

(6) $$\mathbf{A} = \mathbf{a}A.$$

We may regard as characteristic of a coordinate system the three unit-vectors which fall in the direction of the positive axes. These unit-vectors are generally denoted by **i**, **j**, **k**, and are called the *fundamental vectors* of the system in question.

§ 3. Vectorial Algebra.

Relations may be established between vectors, just as between numerical quantities, by various operations ; and these are best defined in such a way that, in special cases, they may be identified with the similarly named arithmetical processes.

We will first define as the *sum of two vectors* a vector which represents the *diagonal of a parallelogram* whose sides

are equal to the two terms, both in magnitude and direction. In order, then, to obtain the sum of two vectors **A** and **B**, we draw the vector **B** from the end of the vector **A**, and join the beginning of **A** to the end of **B** (Fig. 3). The vectorial or, as it is also called, the geometrical sum of the two vectors is expressed by the symbol

$$\mathbf{A} + \mathbf{B}.$$

Fig. 3.

A glance at the figure at once shows us that the *commutative* and *associative* laws hold for vectorial just as for arithmetical addition. That is to say,

(1) $$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A},$$

and, if we denote any arbitrary third vector by **C**, we have

(2) $$(\mathbf{A} + \mathbf{B}) + \mathbf{C} = (\mathbf{A} + \mathbf{C}) + \mathbf{B} = (\mathbf{B} + \mathbf{C}) + \mathbf{A}.$$

If two vectors that are to be added together have the same direction, geometrical addition becomes arithmetical, and the magnitude of the sum is simply equal to the sum of the individual magnitudes.

By the *difference of two vectors*, represented by the symbol

$$\mathbf{A} - \mathbf{B},$$

we understand the sum of the vector **A** and a vector which is *equal and opposite* to **B**.

If A_x, A_y, A_z are the components of a vector **A**, we can regard the latter as the sum of three vectors which fall in the directions of the coordinate axes, and whose magnitudes are A_x, A_y, A_z. Using the symbols for the fundamental vectors of the coordinate system, we obtain the formula

(3) $$\mathbf{A} = \mathbf{i}A_x + \mathbf{j}A_y + \mathbf{k}A_z.$$

Conversely, if a vector **A** can be represented in the form

$$\mathbf{A} = \mathbf{i}S' + \mathbf{j}S'' + \mathbf{k}S''',$$

wherein S', S'' and S''' signify three scalar expressions, we can immediately conclude that S' is the x-component of the vector, S'' the y-, and S''' the z-component.

If we suppose eqn. (3) formed for a second vector **B**, and add it vectorially to the original eqn. (3), we find

(4) $$\mathbf{A} + \mathbf{B} = \mathbf{i}(A_x + B_x) + \mathbf{j}(A_y + B_y) + \mathbf{k}(A_z + B_z).$$
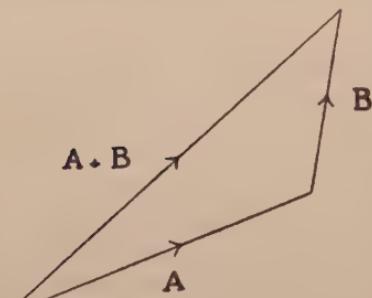
The meaning of this formula is, according to the above, that the *components of the sum* of two vectors are *equal* to the *sums of the components* of the individual vectors.

As regards the *multiplication of vectors*, a distinction has to be made between the so-called internal and the so-called external multiplication. The definition of the *internal* or *scalar product* of two vectors is the scalar obtained by multiplying the *magnitude* of the one vector by the *projection of the other vector upon it*. The scalar product of two vectors **A** and **B** is denoted by the symbol

$$\mathbf{AB}, \text{ or } \mathbf{A} \cdot \mathbf{B}, \text{ or } (\mathbf{AB}).$$

Hence

(5) $$\mathbf{AB} = AB \cos (\mathbf{A}, \mathbf{B}).$$

The scalar product is *positive* or *negative*, according as the angle between the vectors is *acute* or *obtuse*.

It follows from the definition of the scalar product that the *commutative law* holds just as for arithmetical multiplication, *i.e.*,

(6) $$\mathbf{AB} = \mathbf{BA}.$$

We will now further consider the scalar product of a vector **A** and a second vector **D** which, in its turn, is the sum of two vectors **B** and **C**. By making the direction of the vector **A** the x-axis of an otherwise arbitrary coordinate system, we see at once from eqn. (4) that the projection of the vector **D** on the direction of the vector **A** is equal to the sum of the projections of the vectors **B** and **C**. Therefore, according to the definition of the scalar product,

(7) $$\mathbf{A(B + C)} = \mathbf{AB} + \mathbf{AC}.$$

For internal multiplication the *distributive law* holds good as well as the commutative, and so the ordinary rules of arithmetical multiplication can be applied to scalar multiplication.

We see from eqn. (5) that, in two cases, the scalar product assumes an especially simple form, *viz.*, either when the two vectors have the same direction, or when they are normal to each other. In the first case the scalar product is simply equal to the product of the magnitudes, and hence, in particular, the scalar product of any vector and itself is equal to the square of its magnitude, *i.e.*,

(8) $$\mathbf{AA} = A^2.$$

On the other hand,

(9) $$\mathbf{AB} = 0, \text{ when } \mathbf{A} \perp \mathbf{B}.$$

Conversely, the vanishing of the scalar product of two vectors always shows that the vectors are *at right angles* to each other. Applying eqns. (8) and (9) to the fundamental vectors, we obtain the important relations

(10) $$\mathbf{ii} = \mathbf{jj} = \mathbf{kk} = 1,$$

and

(11) $$\mathbf{ij} = \mathbf{jk} = \mathbf{ki} = 0.$$

The *scalar product of two fundamental vectors* is thus always *unity or zero,* according as the two factors are the same or different.

In order to express the scalar product of two vectors in terms of their components, we start from the equation

(12) $$\mathbf{AB} = (\mathbf{i}A_x + \mathbf{j}A_y + \mathbf{k}A_z)(\mathbf{i}B_x + \mathbf{j}B_y' + \mathbf{k}B_z).$$

On multiplying out in accordance with the distributive law, we obtain nine terms on the right-hand side, six of which, however, vanish by eqn. (11). Applying eqn. (10) to the remaining three, we find the simple relation

(13) $$\mathbf{AB} = A_x B_x + A_y B_y + A_z B_z.$$

Dividing this equation by the product AB, and remembering that A_x is equal to $A \cos(\mathbf{A}, x)$ we have, by eqn. (5), the relation

(14) $\begin{cases} \cos(\mathbf{A}, \mathbf{B}) = \cos(\mathbf{A}, x)\cos(\mathbf{B}, x) \\ \qquad + \cos(\mathbf{A}, y)\cos(\mathbf{B}, y) + \cos(\mathbf{A}, z)\cos(\mathbf{B}, z). \end{cases}$

Contrasted with the internal or scalar product, we define

[AB]

B

A

Fig. 4.

the *external* or *vector product* of two vectors **A** and **B** as a *vector,* whose magnitude is equal to the *area* of a *parallelogram* formed by **A** and **B**; that is to say, which contains as many *units of length* as does the parallelogram units of area. Moreover, the sense of its direction is such that, when seen from its extremity, the shortest rotation necessary to carry the first vector in the product into the direction of the second one appears anti-clockwise, insomuch as we draw both
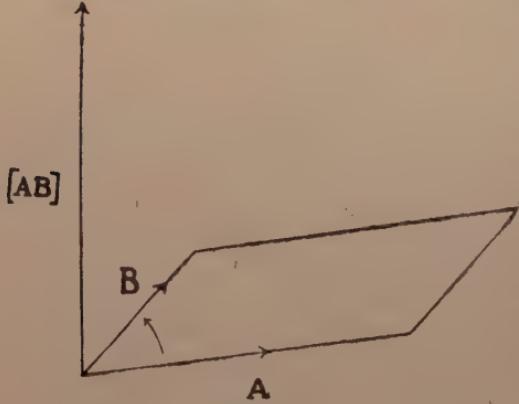
vectors from the same point (Fig. 4). The vector product of two vectors **A** and **B** is denoted by the symbol

$$[\mathbf{AB}].$$

In point of fact the vector product is not really a directed line, but a *directed plane-magnitude* ; for all vectorial operations, however, it is found more convenient to work, instead of with directed plane magnitudes, with the " *auxiliary vectors* " which *supplement* them, and which are erected perpendicularly from the directed plane-magnitudes in the manner shown in the diagram.

It follows from the definition of a vector product that its *magnitude* is given by $AB \sin (\mathbf{A}, \mathbf{B})$, and that its sense is changed on transposing the two vectors in the product ; hence

(15) $$[\mathbf{AB}] = -[\mathbf{BA}].$$

Let us now consider the vector product of a vector **A** and a second vector **D** which, in its turn, is the sum of two vectors, **B** and **C**. For this purpose we may imagine a parallelogram constructed, whose sides are formed by the vectors **B** and **C**, and whose diagonal is hence equal to **D** ; and we will suppose this parallelogram projected on a plane perpendicular to the vector **A**. The projections of the sides and the diagonal are then given by the expressions

$$B \sin (\mathbf{A}, \mathbf{B}) ; \ C \sin (\mathbf{A}, \mathbf{C}) ; \ D \sin (\mathbf{A}, \mathbf{D}).$$

We now imagine the dimensions of this parallelogram to be increased in the ratio $A : 1$, thereby obtaining (as shown in Fig. 5) a parallelogram $OPQR$; the lengths OP, PQ and OQ will then be numerically equal to the magnitudes of the three vector products $[\mathbf{AB}]$, $[\mathbf{AC}]$ and $[\mathbf{AD}]$. Finally, let the parallelogram $OPQR$ be rotated in the plane of the figure through 90°, so that OP is perpendicular to the vector **B** ; then PQ and OQ will naturally be at right angles to the vectors **C** and **D** respectively. Since, however, the plane of the figure is at right angles to the vector **A**, the

Fig. 5.

above three lines will also be normal to **A**. Consequently, after this rotation, the three lines OP, PQ and OQ represent,
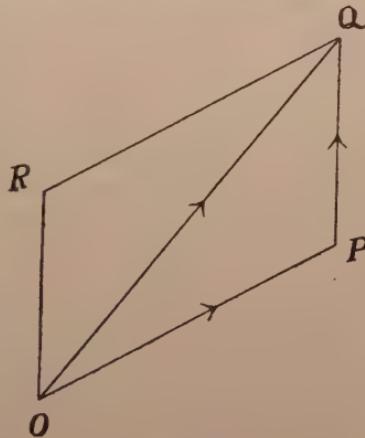
both in magnitude and direction, the three vector products, [**AB**], [**AC**] and [**AD**][1]; and, since the diagonal of a parallelogram is equal to the vector sum of its sides, we obtain the relation

$$[\mathbf{AD}] = [\mathbf{AB}] + [\mathbf{AC}],$$

or, since **D** actually represents the sum of **B** and **C**,

(16) $$[\mathbf{A(B + C)}] = [\mathbf{AB}] + [\mathbf{AC}].$$

Hence the *distributive law* is equally valid in vector and in arithmetical multiplication; from this it follows that vector multiplication can be carried out according to the rules of arithmetical multiplication, only with the difference that, if the order of two factors be reversed, the sign must also be reversed.

The definition of the vector product shows that it assumes a simple form in two cases; *viz.*, when the vectors are at right angles, or when they are parallel. In the first case the magnitude of the vector product is equal to the product of the individual magnitudes; while in the second case we have

(17) $$[\mathbf{AB}] = 0, \text{ when } \mathbf{A} \parallel \mathbf{B}.$$

Conversely, we may always conclude from the vanishing of the vector product of two vectors that the latter have the *same direction.*

Eqn. (17) gives for the *fundamental vectors* the important formulæ

(18) $$[\mathbf{ii}] = [\mathbf{jj}] = [\mathbf{kk}] = 0.$$

If, on the other hand, we form the vector products of two different fundamental vectors, the magnitude of such a product is always unity, and it has the same direction as the third fundamental vector. Consequently, we have

(19) $$\begin{cases} [\mathbf{ij}] = \mathbf{k}; \quad [\mathbf{jk}] = \mathbf{i}; \quad [\mathbf{ki}] = \mathbf{j}; \\ [\mathbf{ji}] = -\mathbf{k}; \ [\mathbf{kj}] = -\mathbf{i}; \ [\mathbf{ik}] = -\mathbf{j}. \end{cases}$$

In order to express the *components of a vector product* in terms of the components of the individual vectors, we have to form, according to the distributive law, the expression

(20) $$[\mathbf{AB}] = [\ (\mathbf{i}A_x + \mathbf{j}A_y + \mathbf{k}A_z)(\mathbf{i}B_x + \mathbf{j}B_y + \mathbf{k}B_z)\ ].$$

[1] In this connection it is unnecessary to speak of a sense of direction.

Of the nine terms obtained by multiplying out three disappear, in accordance with eqn. (18), so that we have

$$[\mathbf{AB}] = [\text{ij}]A_x B_y + [\text{ik}]A_x B_z$$
$$+ [\text{ji}]A_y B_x + [\text{jk}]A_y B_z$$
$$+ [\text{ki}]A_z B_x + [\text{kj}]A_z B_y \,.$$

From this we find, by eqn. (19),

$$[\mathbf{AB}] = \mathbf{k}A_x B_y - \mathbf{j}A_x B_z - \mathbf{k}A_y B_x + \mathbf{i}A_y B_z + \mathbf{j}A_z B_x - \mathbf{i}A_z B_y \,.$$

By arranging the terms to obtain three scalar expressions, each multiplied by $\mathbf{i}$, $\mathbf{j}$ or $\mathbf{k}$, we find for the components of the vector product the values [2]

$$(21) \quad \begin{cases} [\mathbf{AB}]_x = A_y B_z - A_z B_y \,, \\ [\mathbf{AB}]_y = A_z B_x - A_x B_z \,, \\ [\mathbf{AB}]_z = A_x B_y - A_y B_x \,. \end{cases}$$

The definitions of scalar and vector multiplication are sufficient in themselves to make the computation of expressions for *three or more multiplied vectors* purely a matter of calculation. It is immediately clear that an expression of the form $\mathbf{A}(\mathbf{BC})$ represents a vector with the direction of $\mathbf{A}$, since it contains a vector multiplied by a scalar. The expressions $\mathbf{A}(\mathbf{BC})$, $\mathbf{B}(\mathbf{CA})$, $\mathbf{C}(\mathbf{AB})$ are thus vectors which, in general, *differ* from each other both in magnitude and in direction.

The scalar product of one vector and the vector product of two others, *e.g.*, the quantity $\mathbf{A}[\mathbf{BC}]$, must of course also be a *scalar*. It is equal to the area of the parallelogram formed by the vectors $\mathbf{B}$ and $\mathbf{C}$ multiplied by the amount A and, in addition, by the cosine of the angle which $\mathbf{A}$ makes with the normal to the parallelogram. The cosine of this angle, however, is equal to the sine of the angle made by the vector $\mathbf{A}$ with the plane of the parallelogram. In consequence, the scalar product $\mathbf{A}[\mathbf{BC}]$ is equal to the *volume* of the parallelopiped formed by the vectors $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$. Hence we have the important relation

(22) $\mathbf{A}[\mathbf{BC}] = 0$, when $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ are *co-planar* ;

i.e., when the three vectors, drawn from the same point, lie in the same plane.

We can also find the volume of the parallelopiped by multiplying the area of the parallelogram formed by the vectors $\mathbf{C}$ and $\mathbf{A}$ by the quantity B and, in addition, by the

[2] From the first of the three eqns. (21) we obtain the second, and from this the third, by a *cyclic* interchange of the indices x, y, z—*i.e.*, in the order x, y, z, x, *etc.* ; x becomes y, y becomes z, and z is replaced by x again.

sine of the angle which **B** makes with the plane of the parallelogram. Analogous considerations hold for the parallelogram formed by the vectors **A** and **B**; and so, through cyclic interchange,[3] we obtain the relation

$$(23) \qquad \mathbf{A[BC]} = \mathbf{B[CA]} = \mathbf{C[AB]}.$$

The double vector product $\left[\mathbf{A[BC]}\right]$ must undoubtedly also be a vector. If we denote the vector $[\mathbf{BC}]$ by **E**, we have, by eqn. (21),

$$\left[\mathbf{A[BC]}\right]_x = A_y E_z - A_z E_y = A_y(B_z C_y - B_y C_z) \\ - A_z(B_x C_x - B_x C_z).$$

On adding to this equation the identity

$$0 = A_x B_x C_x - A_x B_x C_x,$$

we obtain

$$\left[\mathbf{A[BC]}\right]_x = B_x(A_x C_x + A_y C_y + A_z C_z) \\ - C_x(A_x B_x + A_y B_y + A_z B_z).$$

Similar equations hold for the y- and z-components of the double vector product. Hence, taking into account eqn. (13), we find

$$(24) \qquad \left[\mathbf{A[BC]}\right] = \mathbf{B(CA)} - \mathbf{C(AB)}.$$

The *differentiation of a vector with respect to a scalar variable* can always be reduced to vector subtraction, if the differential coefficient be regarded as the limit of the rate of change of the vector function. Thus, it follows from eqn. (3) that the differential coefficient of a vector with respect to time is given by

$$(25) \qquad \frac{d\mathbf{A}}{dt} = \mathbf{i}\frac{dA_x}{dt} + \mathbf{j}\ \frac{dA_y}{dt} + \mathbf{k}\frac{dA_z}{dt},$$

[3] The correctness of the *signs* in eqn. (23), where there is a cyclic interchange of the three vectors **A**, **B**, **C**, may be demonstrated as follows. Let us suppose that the plane containing the vectors **B** and **C** is horizontal, such as the surface of a table, and that **B** and **C** are so arranged in this plane that, when seen from above, an anti-clockwise rotation serves to convey **B** into the direction of **C**. Two cases are now to be distinguished: in the first the vector $[\mathbf{BC}]$ makes an acute angle with **A**, and the scalar product $\mathbf{A[BC]}$ is positive; in the second the angle is obtuse, and $\mathbf{A[BC]}$ negative. In the first case, the vector **A** must lie in an upward direction from the horizontal plane, and inspection at once shows us that the shortest rotation serving to transfer **C** into the direction of **A** is anti-clockwise, when seen from the vector **B**. In that case, the angle between the vectors $[\mathbf{CA}]$ and **B** must also be acute, and therefore the scalar product $\mathbf{B[CA]}$ positive; *i.e.*, the products $\mathbf{A[BC]}$ and $\mathbf{B[CA]}$ have the same sign. The same result can easily be shown to hold also in the second case, where **A** makes an obtuse angle with the vector $[\mathbf{BC}]$. (We can most readily illustrate the situation if we lay two differently coloured pencils on a table, with their ends together, so that they point in different directions, and then hold a third pencil or a penholder so that it slopes in an arbitrary upward direction from their point of contact.)

provided that the essential condition is fulfilled, that the position of the coordinate system itself does not change with time.

From eqns. (4) and (25), we have

(26) $$\frac{d(\mathbf{A} + \mathbf{B})}{dt} = \frac{d\mathbf{A}}{dt} + \frac{d\mathbf{B}}{dt}.$$

Similarly it follows from eqn. (13) that

(27) $$\frac{d}{dt}(\mathbf{AB}) = \mathbf{A}\frac{d\mathbf{B}}{dt} + \mathbf{B}\frac{d\mathbf{A}}{dt}.$$

From eqn. (21) we obtain the rule for the differentiation of a vector product :

(28) $$\frac{d}{dt}[\mathbf{AB}] = \left[\mathbf{A}\frac{d\mathbf{B}}{dt}\right] + \left[\frac{d\mathbf{A}}{dt}\mathbf{B}\right].$$

On the other hand, we can always put [§ 2 eqn. (6)]

(29) $$\mathbf{A} = \mathbf{a}A,$$

where $\mathbf{a}$ is the *unit-vector* falling in the direction of a vector $\mathbf{A}$, so that

(30) $$\frac{d\mathbf{A}}{dt} = \mathbf{a}\frac{dA}{dt} + A\frac{d\mathbf{a}}{dt}.$$

Now, by eqn. (8),

$$\mathbf{aa} = 1,$$

and hence, by eqn. (27),

$$\mathbf{a}\frac{d\mathbf{a}}{dt} = 0.$$

Therefore, by eqn. (9), since the directions of the vectors $\mathbf{a}$ and $\mathbf{A}$ are the same,

(31) $$\frac{d\mathbf{a}}{dt} \perp \mathbf{A}.$$

If, now, we denote by $\mathbf{a}_1$ and $\mathbf{a}_2$ the values which a unit-vector (drawn from any given point) has at the beginning and end of an interval dt, the values $\mathbf{a}_1$, $\mathbf{a}_2$ and $d\mathbf{a}$ must make up a triangle (Fig. 6). If we measure the angle through which the vector $\mathbf{a}$ turns in the interval dt in circular measure,

Fig. 6.

and call it $d\varphi$, then the magnitude of $d\mathbf{a}$ is given by $d\varphi$. We may accordingly state eqn. (30) in the form that the *differential coefficient of a vector with respect to time* can
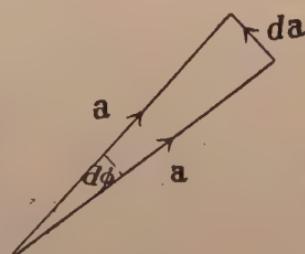
always be *resolved into two components*; one of these lies *in the direction of the vector* and is of magnitude dA/dt, while the other is *at right angles to it* and of magnitude $A \, d\varphi/dt$, where $d\varphi$ is the angle through which the vector **A** *turns* in the interval dt. Both components lie in the surface traversed by the vector during that interval, this surface being regarded as a plane.

§ 4. The Equations of Motion of a Particle.

The connection existing between the two vectors of force and velocity, which are essential for the consideration of motion, finds its expression in *Newton's Second Law of Motion*. According to this law, the *force vector* for every body is always *proportional* to the *time-rate of change of the velocity vector*; the proportionality factor thus appears as a *constant* characteristic of the body, and is termed the *mass* of the body.[1]

If we regard velocity and force as vectors, denoting them respectively by **v** and **K**, and if we call the mass of a body m, we can then express Newton's Second Law of Motion in the vectorial form

$$(1) \qquad \mathbf{K} = m\frac{d\mathbf{v}}{dt}.$$

The differential coefficient of the velocity vector with respect to time is termed the *acceleration*, and is itself a vector. To simplify the following investigations we shall assume that the body moves as if its *whole mass* were *concentrated at a single point*. The following considerations are therefore based on an imaginary so-called " *particle*."[2]

[1] Newton's statement of the Second Law of Motion is : " Mutationem motus proportionalem esse vi motrici impressæ et fieri secundum lineam rectam qua vis illa imprimitur " (" Change of motion is proportional to the impressed force and takes place in the direction in which that force is impressed "). Newton adds to this, as a second definition, the statement : " Quantitas motus est mensura eius orta ex velocitate et quantitate materiæ coniunctim " (" The quantity of motion is measured by the product of the velocity and the mass "). Theoretical physics takes as unit of mass the gram, *i.e.*, the thousandth part of a standard mass which is preserved at Sèvres (near Paris), and which, with very fair accuracy, is equal to the mass of a cubic decimeter of water at 4° C. The unit of length is the centimetre, *i.e.*, the hundredth part of a standard measure of length preserved at Sèvres. As unit of time we have the second, *i.e.*, the $24 \times 60 \times 60$th part of the mean solar day. From these three units we can readily derive the units of velocity and acceleration, as well as that of force, which is termed a *dyne*.

[2] The imaginary concentration of the mass in a single point greatly simplifies the problem, especially when the acting force is a space function, and so changes from place to place, in which case consideration of the distribution of the body

The position of the moving particle at any moment can be determined by a *radius-vector* **r** which is drawn from any given *origin* to the particle, and whose components are the coordinates x, y, z of the latter.

Let the values of the radius-vector at times t and $(t + dt)$ be $\mathbf{r}_1$ and $\mathbf{r}_2$; then

(2) $$\mathbf{r}_2 = \mathbf{r}_1 + \frac{d\mathbf{r}}{dt}\,dt\,.$$

On the other hand, if $d\mathbf{s}$ be the *element of path* traversed in the element of time,

(3) $$\mathbf{r}_2 = \mathbf{r}_1 + d\mathbf{s}$$

(Fig. 7). Now, by the definition of velocity,

(4) $$d\mathbf{s} = \mathbf{v}\,dt\;;$$

and it follows from eqns. (2) and (3) that

(5) $$\mathbf{v} = \frac{d\mathbf{r}}{dt}\,.$$
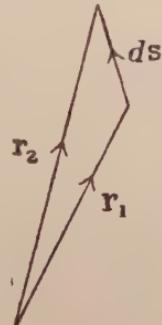
FIG. 7.

The *acceleration* vector **b** is by definition equal to the time-rate of change of the velocity vector, so that

(6) $$\mathbf{b} = \frac{d\mathbf{v}}{dt}\,,$$

or,

(7) $$\mathbf{b} = \frac{d^2\mathbf{r}}{dt^2}\,.$$

By writing these equations in *analytical* form, we find the relations

(8) $$v_x = \frac{dx}{dt},\; v_y = \frac{dy}{dt},\; v_z = \frac{dz}{dt}\;;$$

and

(9) $$b_x = \frac{d^2x}{dt^2},\;\; b_y = \frac{d^2y}{dt^2},\;\; b_z = \frac{d^2z}{dt^2}\,.$$

Taking these relations into account, we may resolve eqn. (1) into the analytical *equations of motion*

(10) $$X = m\frac{d^2x}{dt^2},\;\; Y = m\frac{d^2y}{dt^2},\;\; Z = m\frac{d^2z}{dt^2}\,,$$

in which X, Y, Z represent the components of the force.

Since acceleration is the differential coefficient of velocity

in space would essentially complicate the problem. Moreover, this imaginary concentration of the mass has the advantage that we can disregard all forces which might act within a body, all changes of shape, and all eventual rotations.

with respect to time, it may be resolved [according to § 3 eqn. (30)] into two components at right angles, one of which lies in the direction of the velocity and is termed the *tangential acceleration*, while the other component perpendicular to it is called the *normal acceleration*. It follows from the concluding remarks of § 3 that the value of the tangential acceleration (b_t) is given by

$$(11) \qquad b_t = \frac{dv}{dt},$$

and that of the normal acceleration by

$$(12) \qquad b_n = v\frac{d\varphi}{dt},$$

where $d\varphi$ is the angle through which the velocity vector turns in the interval dt. If AB is the path-element described in the interval dt, $d\varphi$ is equal to the angle included between the tangents to the path at the points A and B—the so-called " total curvature " of the arc AB. Now, by an elementary rule of differential geometry (which we may assume to be known), the arc AB is equal to the product of this angle (measured in radians) and the value of the *radius of curvature* for that particular portion of the curve. Thus, if ϱ is the value of the radius of curvature, we have

$$ds = \varrho\, d\varphi,$$

or, dividing through by dt,

$$v = \varrho\,\frac{d\varphi}{dt}.$$

Hence,

$$(13) \qquad b_n = \frac{v^2}{\varrho}.$$

If lines are drawn from the point A representing the directions of the velocity vector at the points A and B, a plane is thereby determined which is called the *osculating plane* of the arc AB. Consequently, the vectors $\mathbf{v}$ and ($\mathbf{v} + d\mathbf{v}$), as well as the vector $d\mathbf{v}/dt$, *i.e.*, the acceleration vector with both its components, all lie in this plane.[3] In the case of *circular motion* ϱ is simply equal to the radius of the circle, and the normal component of the acceleration is then termed the *centripetal acceleration*.[4]

[3] The direction normal to the osculating plane is termed the *binormal*. The acceleration component in this direction is thus always zero.

[4] If the *period of revolution* be T, eqn. (13) can be stated, for the case of circular motion, in the form

$$b_n = \frac{4\pi^2 r}{T^2}.$$

The product of the mass and the velocity vector is called the *momentum*. If $\mathbf{G}$ represents the momentum, we have

$$(14) \qquad\qquad \mathbf{G} = m\mathbf{v},$$

and consequently

$$(15) \qquad\qquad \mathbf{K} = \frac{d\mathbf{G}}{dt}.$$

The product $\mathbf{K}\, dt$ (which is likewise also a vector) is also called the *impulse* acting during the interval dt. Hence eqn. (15) may be expressed in the form that the increase of momentum is equal to the impulse.

An important corollary to Newton's Second Law of Motion is the principle of the simple *superposition* of several forces, when they act simultaneously on a particle. Thus, if $\mathbf{b}_1$ and $\mathbf{b}_2$ are the accelerations produced by the separate action of each of two forces on a particle, the *actual acceleration* produced by the *simultaneous* action of the forces is given by $\mathbf{b}$ where, in accordance with the principle of superposition,

$$(16) \qquad\qquad \mathbf{b} = \mathbf{b}_1 + \mathbf{b}_2.$$

If this equation be multiplied with the mass of the particle, the products $m\mathbf{b}_1$ and $m\mathbf{b}_2$ give the two acting forces $\mathbf{K}_1$ and $\mathbf{K}_2$. Hence we find the actual acceleration given by the relation

$$(17) \qquad\qquad m\frac{d\mathbf{v}}{dt} = \mathbf{K}_1 + \mathbf{K}_2.$$

A particle subjected to the simultaneous action of several forces moves, therefore, as though a single force were acting on it, which is equal to the vector sum of the separate forces. This force is called the *resultant force*; and, since it is found by the construction of a parallelogram, according to the definition of vectorial addition, the principle of superposition here involved is also known as the *principle of the parallelogram of forces*.

The theorem of the parallelogram of forces in its application to static problems is due to *Stevin*,[5] whilst its general dynamical validity was first demonstrated by *Newton*. The resolution of acceleration into a tangential and a normal component was carried out before Newton by *Huygens*, in 1673. The equations of motion of a particle were first derived by *Euler*, in 1765, from Newton's Second Law of Motion.

[5] To Stevin is due the law that three forces acting at a point are in equilibrium when the directed lines representing them can be arranged to form a *triangle*.

§ 5. Projectile Motion and Free Fall.

The *motion of a body in the gravitational field of the earth* is given by the differential equation

$$(1) \qquad \frac{d\mathbf{v}}{dt} = \mathbf{g},$$

where $\mathbf{g}$, the *acceleration due to gravity*, is a vector with a direction vertically downwards, having a value for a mean geographical latitude (of $45°$) of about 981 cm. sec.$^{-2}$

By integrating eqn. (1) we obtain the relation

$$(2) \qquad \mathbf{v} = \mathbf{g}t + \mathbf{u},$$

where $\mathbf{u}$, the vector constant of integration, represents the value of the velocity at a time ($t = 0$). Motion in the earth's gravitational field, when the initial velocity is not zero, is called *projectile motion*. The vector $\mathbf{u}$ determines the *direction*, and its magnitude the *velocity of projection*. The angle made by the direction of projection with the horizontal is termed the *angle of elevation*.

Now, it follows from eqn. (2) that the projected body does not depart from the plane which is determined by the vectors $\mathbf{u}$ and $\mathbf{g}$, that is to say, from the vertical plane which contains the direction of projection. Let this be the x-z-plane of a plane coordinate system with a horizontal x-axis, and a z-axis drawn vertically upwards. Then, if the angle of elevation be α,

$$(3) \qquad u_x = u \cos \alpha, \quad u_z = u \sin \alpha.$$

Since the vector $\mathbf{g}$ lies in the direction of the negative z-axis, eqn. (2) may be analytically expressed in the form

$$(4) \qquad \frac{dx}{dt} = u \cos \alpha, \quad \frac{dz}{dt} = -gt + u \sin \alpha.$$

Integrating, we have

$$(5) \qquad x = ut \cos \alpha, \quad z = -\frac{gt^2}{2} + ut \sin \alpha,$$

the integration constants vanishing, because the system of coordinates was so chosen that for $t = 0$ both x and z are zero.

Eliminating the time from eqn. (5), we obtain the equation

$$(6) \qquad z = -\frac{g}{2} \frac{x^2}{u^2 \cos^2 \alpha} + x \tan \alpha,$$

which is the equation of the *trajectory*; it represents a

parabola whose axis is parallel to the z-axis and thus vertical, since z only occurs in the first power.[1] The parabola cuts the x-axis not only at the origin, but also at a second point, the distance of which from the origin (OA in Fig. 8) is called the *range*. We may find the range (a) by putting $z = 0$ in eqn. (6) ; then [2]

$$(7) \qquad a = \frac{2u^2 \tan \alpha \cos^2 \alpha}{g} = \frac{u^2 \sin 2\alpha}{g}.$$

The z-coordinate of the vertex of the trajectory is known as the *height of ascent*, and the time that elapses before this point is reached is called the *time of ascent*. The latter can be ascertained from the fact that, at the vertex, the vertical velocity (or dz/dt) must vanish. If t' be the time of ascent, we find from eqn. (4)

Fig. 8.

$$(8) \qquad t' = \frac{u}{g} \sin \alpha.$$
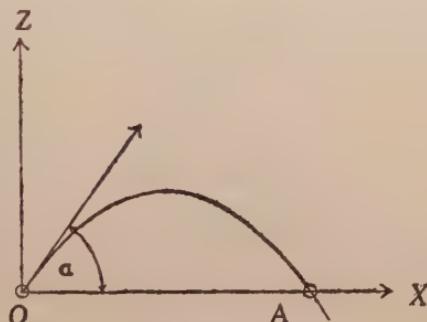
Inserting this value in the second of the two eqns. (5), we have for the height of ascent, which may be denoted by z',

$$z' = -\frac{u^2 \sin^2 \alpha}{2g} + \frac{u^2 \sin^2 \alpha}{g},$$

or,

$$(9) \qquad z' = \frac{u^2 \sin^2 \alpha}{2g} = \frac{gt'^2}{2}.$$

From eqn. (7) we can determine the angle of elevation at which a body must be projected with a given velocity in order to reach a target at a horizontal distance a ; for we must have

$$(10) \qquad \sin 2\alpha = \frac{ag}{u^2}.$$

Now, if a is greater than u^2/g, $\sin 2\alpha$ is greater than

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[1] The equation of a parabola whose axis coincides with the z-axis, and whose vertex lies on the x-axis, is, as is well known,

$$x^2 = 2pz,$$

where p is the semi-latus rectum.   If the axes of x and z are displaced parallel to themselves, the equation still remains linear with respect to z.

[2] Making use of the well-known formula $\sin 2a = 2 \sin a \cos a$.

unity, and no real solution is possible; the target, in fact, lies too far off for us to reach it with the given velocity. On the other hand, if a is less than u^2/g, $\sin 2\alpha$ is less than unity; and, since supplementary angles have the same sine, there are two possible solutions, α and $(90° - \alpha)$.

With the given velocity of projection, the target can therefore be reached with either of *two different angles of elevation complementary to each other*, by means of a *steeper* or a *flatter* parabola (Fig. 9). If, in particular, a is equal to u^2/g, the two parabolas coincide, and there is only one value for the angle of elevation, *viz.*, 45°. Conversely, it follows from eqn. (7) that the *maximum range* is attained with an angle of elevation of 45°.
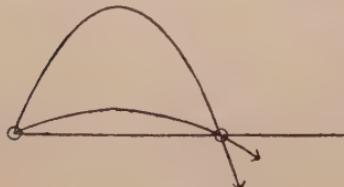
Fig. 9.

When the velocity of projection vanishes, we have, as a special case of projectile motion, *free fall*. We then obtain from eqns. (4) and (5) the following expressions for the velocity (v) and the distance of fall (h):

(11) $$v = gt, \quad h = \tfrac{1}{2} gt^2,$$

giving as a relation between v and h

(12) $$v = \sqrt{2gh}.$$

The elementary laws both of free fall and of projectile motion had already been derived by *Galileo*, about 1590, and were published in his "*Discorsi*," which appeared in 1638.

Finally, let us briefly consider the *influence of a resisting medium on the motion of free fall*. Experience shows that, for moderate velocities, the *resistance* can be assumed to be *proportional to the velocity* and directly opposed to it; the proportionality factor is called the *coefficient of frictional resistance*. For a particle which falls in a resisting medium we have, accordingly, the equation

(13) $$\frac{dv}{dt} = g - kv,$$

where k is the quotient of the coefficient of frictional resistance and the mass of the falling body.

Integrating eqn. (13), we find

$$\int \frac{dv}{g - kv} = t + C,$$

or

$$\log (g - kv) = - kt + \log C',$$

or

$$g - kv = C'e^{-kt},$$

C and C' being constants of integration.   Since $v = 0$ when $t = 0$ for the case of free fall, C' is simply equal to g, and so

(14) $$v = \frac{g}{k} (1 - e^{-kt}).$$

Hence, with increasing values of t, the velocity approaches the *constant limit* g/k.   Since k is the quotient of the coefficient of frictional resistance and the mass, the constant velocity of fall is, roughly speaking, more quickly attained, the larger the resistance and the lighter the falling body.

§ 6. Motion of a Pendulum.

The simplest instance of a so-called *pendulum* consists in a heavy body, assumed to be a particle, suspended by a thread of negligible mass.   If it be displaced from its position of equilibrium, in which the thread is vertical, and then freed without any velocity being imparted to it, it describes a so-called *pendulum motion*.   The angle which the direction of the thread makes with the vertical is called the *angular displacement*: measured in radians, it may be denoted by φ. For *small angular displacements* (and we shall only consider these in what follows), the velocity of the motion of a pendulum may be expressed by

(1) $$v = l\frac{d\varphi}{dt},$$

where l is the length of the pendulum.

The *force of gravity* acting on the pendulum may be resolved into two components, one in the direction of the thread, and the other at right angles to it ; the value of the former is

Fig. 10.

mg cos φ and of the latter $-mg$ sin φ (*cf.* Fig. 10).   The tension of the thread annuls the effect of the first component, so that only the second remains to be considered.   Hence the equation of motion of the pendulum takes the form

(2) $$l\frac{d^2\varphi}{dt^2} = - g \sin \varphi,$$

or, since $\sin \varphi$ can be put equal to φ for the small angular displacements in question,

$$(3) \qquad \frac{d^2\varphi}{dt^2} = -\frac{g}{l}\,\varphi\,.$$

Since the second derivative of a *sine function* is equal to, but of opposite sign from the latter,[1] it is easy to see that the solution of the differential equation (3) is given by

$$(4) \qquad \varphi = \varphi^* \sin\left(t\sqrt{\frac{g}{l}}\right),$$

where φ^* is a constant. For, by differentiating eqn. (4), we obtain

$$(5) \qquad \frac{d\varphi}{dt} = \varphi^* \sqrt{\frac{g}{l}} \cos\left(t\sqrt{\frac{g}{l}}\right),$$

whence, differentiating a second time,

$$(6) \qquad \frac{d^2\varphi}{dt^2} = -\varphi^*\frac{g}{l}\sin\left(t\sqrt{\frac{g}{l}}\right) = -\frac{g}{l}\,\varphi\,.$$

It is true that the argument of the sine function might be increased by the addition of a constant, but this constant vanishes if we choose the zero of time so that the value of the angular displacement is then also zero.

Thus the angular displacement varies in value periodically, and also changes from side to side alternately, for the sign of φ is alternately positive and negative, so that the pendulum executes an *oscillation* about its position of equilibrium. The maximum value of φ is given by the constant φ^*, and this is consequently termed the *amplitude*: φ has then an extreme value, and therefore $d\varphi/dt$ and the velocity are both zero. On the other hand, the velocity reaches a maximum when $d^2\varphi/dt^2$ becomes zero, and, according to eqn. (3), this is the case when φ itself vanishes: *i.e.*, the pendulum passes through its position of equilibrium with maximum velocity.

The *time of a complete oscillation*, known as the *period*, is determined by a time τ, whose value follows from eqn. (4):

$$\tau\sqrt{\frac{g}{l}} = 2\pi\,,$$

or,

$$(7) \qquad \tau = 2\pi\sqrt{\frac{l}{g}}\,.$$

---

[1] The same remark, of course, applies to a cosine function.

Hence, as was discovered by *Galileo*, the *period is proportional to the square root of the length of the pendulum.* On the other hand, it is independent of the mass and, provided only oscillations of small angular displacement are considered, also of the amplitude. The oscillations of pendulums of equal length, but of any weight whatsoever, are " *isochronous.*"

§ 7. Moment of Force, and Angular Momentum.

By vectorially multiplying the radius-vector **r**, drawn from the origin, by the force, velocity, or momentum, we arrive at three conceptions which are of great importance in mechanics. The vector product

(1) $$[\mathbf{rK}] = \mathbf{M}$$

is defined as the *moment of the force.* The vector product

(2) $$[\mathbf{rv}] = \mathbf{f}$$

is called the *moment of the velocity,*[1] and the vector product

(3) $$[\mathbf{rG}] = \mathbf{U}$$

the *moment of momentum,* or the *angular momentum.* All these three quantities are referred to the point from which the radius-vector **r** is drawn.

As regards the *moment of force*, its *magnitude* is equal to the *product of the force* and the *perpendicular* dropped from the origin *on to the direction of the force.* It follows from the *definition of a vector product* that the vector of the moment is perpendicular to the plane determined by the origin, the point of application of the force, and the force vector drawn from the latter point. Moreover, the sense of the vector is such that the turning effect which the force would produce, were the point of application and the origin rigidly connected, appears anti-clockwise when seen from the extremity of the vector (Fig. 11). The components of the moment of force with respect to any given system of coordinates are [according to § 3 eqn. (21)] determined by the formulæ

(4) $$\begin{cases} M_x = yZ - zY, \\ M_y = zX - xZ, \\ M_z = xY - yX. \end{cases}$$

The magnitude of the *moment of velocity* is equal to the

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[1] It is a matter of definition whether the moment of velocity signifies the whole, or half, of the vector product of **r** and **v**.

area of a parallelogram formed by the vectors **r** and **v**. It is obvious that the value of this is *twice* that of the area of the surface described by the radius-vector in the interval dt and referred to the unit of time (Fig. 12). The vector of the moment of velocity stands perpendicular to the shaded surface, and its sense is such that, when seen from its extremity, the motion of the radius-vector appears anti-
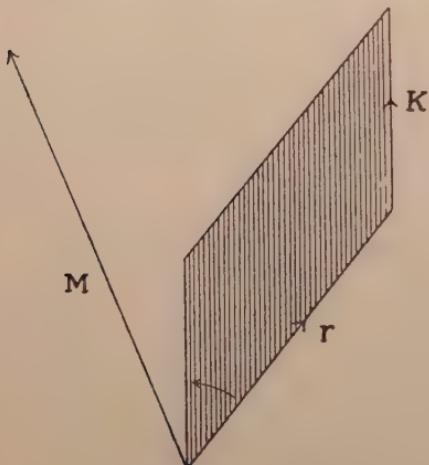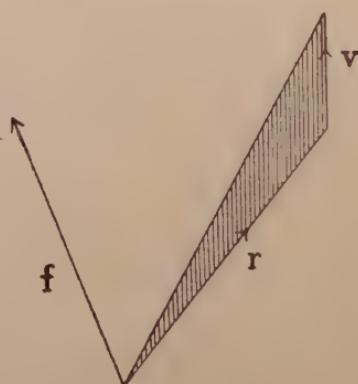
FIG. 11.   FIG. 12.

clockwise. According to § 3 eqn. (21), the following relations hold for the components of the vector :—

(5)
$$\begin{cases} f_x = y\,\dfrac{dz}{dt} - z\,\dfrac{dy}{dt}, \\[2mm] f_y = z\,\dfrac{dx}{dt} - x\,\dfrac{dz}{dt}, \\[2mm] f_z = x\,\dfrac{dy}{dt} - y\,\dfrac{dx}{dt}. \end{cases}$$

The *angular momentum* is obtained simply by multiplying the moment of velocity by the mass.

Now, if we multiply the expression for Newton's Second Law of Motion as given in § 4 eqn. (1) vectorially by **r**, we find the relation

(6)
$$\mathbf{M} = m \left[\mathbf{r}\,\frac{d\mathbf{v}}{dt} \right].$$

But, by the rule for the differentiation of a vector product [§ 3 eqn. (28)],

(7) $$\left[\mathbf{r}\,\frac{d\mathbf{v}}{dt}\right] = \frac{d}{dt}[\mathbf{r}\mathbf{v}] - \left[\frac{d\mathbf{r}}{dt}\,\mathbf{v}\right];$$

and, by § 4 eqn. (5), the vectors $\mathbf{v}$ and $d\mathbf{r}/dt$ are identical. Hence the final vector product in eqn. (7) vanishes, and eqn. (6) assumes the simple form

(8) $$\mathbf{M} = \frac{d\mathbf{U}}{dt}.$$

The *moment of force* is the *time-rate of change of the angular momentum* : the two quantities stand in the same relation to each other as force and momentum [*cf.* § 4 eqn. (15)].

The *moment of force vanishes* when the vectors $\mathbf{r}$ and $\mathbf{K}$ have the same direction, *i.e.*, when the force is permanently directed towards the point from which the radius-vector is drawn. In that case, the *vector of the moment of velocity remains constant both in magnitude and direction*; and we derive the important theorem, first propounded by *Newton*, that, when a body is acted on by a *force* which is *always directed towards the same point*, the motion of the body *continues in one and the same plane*, and its *moment of velocity about that point does not vary*. Conversely, if the moment of velocity with respect to a certain point is constant both in magnitude and direction, we can always conclude from eqns. (1) and (8) that the force is directed towards this point.

§ 8. Planetary Motions.

Newton himself has given the finest example of the extraordinary fruitfulness of his principles of mechanics by his deduction, with their help, of the law of universal *gravitation* from *Kepler's laws*, which describe the course of *planetary motions*. We shall carry out this deduction in the following paragraphs by making use of the simple methods of vectorial algebra.

Kepler's first law states that the radius-vector, drawn from the sun to a planet, *describes equal areas in equal times*. The second law states that the *orbits of the planets are ellipses* with the *sun* at one of the *foci*. These two laws were established in 1609, and in 1619 Kepler added a third, which states that the *squares of the periods of revolution* of the different planets are proportional to the *cubes of the major axes* of their respective orbits.

It follows at once from Kepler's first law, in accordance with what was said at the end of § 7, that the *acceleration*

of a planet in its curvilinear orbit must *always be directed towards the sun.*

In order to find a mathematical expression for Kepler's second law, we will denote the vector connecting the sun with the second focus by $2\mathbf{e}$; then the length of the radius-vector drawn from the second focus to the planet is given by

$$(1) \qquad r' = \sqrt{r^2 + 4e^2 - 4\mathbf{er}},$$

where $\mathbf{er}$, as usual, indicates the scalar product of the two vectors. Now, by the definition of an ellipse, we have

$$(2) \qquad r + r' = 2a,$$

where $2a$ is the major axis. Inserting for r' the value given by eqn. (1) and squaring, we find

$$r^2 + 4e^2 - 4\mathbf{er} = 4a^2 + r^2 - 4ar,$$

or, if we write $2b$ for the minor axis,

$$(3) \qquad \mathbf{er} = ar - b^2$$

(according to the well-known property of an ellipse $a^2 = b^2 + e^2$). Differentiating eqn. (3) with respect to the time, we obtain, since $\mathbf{e}$, a and b are all constant,

$$(4) \qquad \mathbf{e}\frac{d\mathbf{r}}{dt} = a\frac{dr}{dt};$$

whence, by a second differentiation,

$$(5) \qquad \mathbf{e}\frac{d^2\mathbf{r}}{dt^2} = a\frac{d^2\mathbf{r}}{dt^2}.$$

The left-hand side of this equation represents the scalar product of the vector $\mathbf{e}$ and the acceleration vector which, as we already know, is directed toward the sun. Hence, writing ψ for the angle included between the vector $\mathbf{e}$ and the radius-vector $\mathbf{r}$ (whose sense is opposite to that of the acceleration—*cf.* Fig. 13), and denoting the magnitude of the acceleration by γ,[1] we find

FIG. 13.

$$(6) \qquad -\gamma = \frac{a}{e\cos\psi}\frac{d^2r}{dt^2}.$$

Now, by eqn. (3),

$$(7) \qquad e\cos\psi = a - \frac{b^2}{r},$$

[1] We cannot here make use of b, which ordinarily represents the acceleration, for it is also customary to denote by b the semi-minor axis of an ellipse.
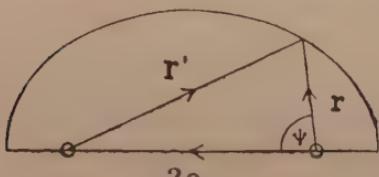
and, differentiating this with respect to the time,

$$(8) \qquad -e \sin \psi \frac{d\psi}{dt} = \frac{b^2}{r^2} \frac{dr}{dt} .$$

But, since the moment of velocity is constant, if we denote it by C we obtain

$$(9) \qquad r^2 \frac{d\psi}{dt} = C ,$$

and from eqn. (8)

$$(10) \qquad \frac{dr}{dt} = -\frac{Ce}{b^2} \sin \psi .$$

Differentiating once more with respect to the time, we have

$$(11) \qquad \frac{d^2r}{dt^2} = -\frac{Ce}{b^2} \cos \psi \frac{d\psi}{dt} ,$$

or, by eqn. (9),

$$(12) \qquad \frac{d^2r}{dt^2} = -\frac{C^2e}{b^2r^2} \cos \psi .$$

Hence, substituting for d^2r/dt^2 in eqn. (6),

$$(13) \qquad \gamma = \frac{aC^2}{b^2r^2} .$$

In order that we may apply Kepler's third law, we will now introduce the *period of revolution* into eqn. (13). The simplest way to do this is by making use of the relation according to which the area of the ellipse is equal to the constant rate of description of equal areas multiplied by the period, *viz.*:

$$(14) \qquad \pi ab \doteq \frac{CT}{2} .$$

Substituting for b in eqn. (13), we have

$$(15) \qquad \gamma = \frac{4\pi^2a^3}{r^2T^2} .$$

Finally, taking into account the constant ratio between a^3 and T^2 given by Kepler's third law for all the planets, and calling it C', we obtain the simple relation

$$(16) \qquad \gamma = \frac{4\pi^2C'}{r^2} .$$

Thus we have deduced from Kepler's laws, in a purely mathematical manner, the important result that the *accelera-*

tion of every planet is always directed towards the sun, and is completely independent of all individual properties of the planet; it is, in fact, *only a function of the distance of the planet from the sun*, and is *inversely proportional to the square of this distance*.

Newton discovered, moreover, that the motions of the *satellites* of Jupiter and Saturn round their respective planets are governed by the same laws as the motions of the planets round the sun. He was also able to prove that the *motion of the moon round the earth* could be explained with the help of the *earth's gravitational field*, on the assumption that the intensity of the latter varies inversely with the square of the distance from the centre of the earth.[2] He accordingly considered it a justifiable conclusion that the forces exerted by the sun on the planets, by Jupiter and Saturn on their respective satellites, and by the earth on the moon and on bodies on its own surface, are essentially identical, and that they all represent only special cases of a *universal force of gravitation*, which is exerted by every single body in the universe on every other body.

Since the constant C' contains no quantity in any way connected with the individual properties of a planet, it follows from eqn. (16) that the acceleration of a planet, due to the sun, is completely independent of the planet's mass. This conclusion is borne out by manifestations of the earth's field of gravity, for the acceleration of all bodies falling freely is equally large and independent of their mass. If, however, the *acceleration* produced in a body A of mass m_1, in consequence of a gravitational action exerted on it, is, in general, *independent of its own mass*, then the force of attraction exerted on the body A by a second body B at a distance r can only be of the form

$$(17) \qquad K = \frac{m_1 S_1}{r^2},$$

in which S_1 must be independent of both r and m_1.

Now the force exerted by the body B on the body A, in virtue of the general attraction of masses, must always be accompanied by an opposite and equally large force

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[2] For, if this is the case, the acceleration due to the earth's field at the distance of the moon must be $1/60 \times 60$, or $1/3600$th of that at the earth's surface, seeing that the mean distance of the moon from the earth is some sixty times the radius of the latter. Newton actually found that the moon's acceleration with respect to the earth, calculated from Huygens' theorems of centripetal acceleration, is $1/3600$th part of that possessed by a body falling freely at the earth's surface.

which is exerted by A on B.   Hence, denoting the mass of the body B by m_2, we must have

$$(18) \qquad K = \frac{m_2 S_2}{r^2},$$

where S_2 is also independent of both r and m_2.[3]   But eqns. (17) and (18) can only be simultaneously satisfied if the force fulfils the relation

$$(19) \qquad K = \kappa \frac{m_1 m_2}{r^2},$$

where κ is a *universal constant* which is independent of both masses, of r, and of all individual properties of either body. κ is called the *constant of gravitation*, and its value is—

$$(20) \qquad 6\cdot 68 \times 10^{-8} \text{ gram}^{-1} \text{ cm.}^3 \text{ sec.}^{-2}.$$

Eqn. (19) is an expression of *Newton's Law of Gravitation,* which states that any two bodies attract each other with a force which is directly proportional to the product of their masses, and inversely proportional to the square of the distance between them.

§ 9. Transformation of Vector Components.

In changing from one system of coordinates to another, the *components* of a vector must be *transformed.*   In finding the relations that exist between the new and the old components, we will denote the fundamental vectors of the first system of coordinates by $\mathbf{i}, \mathbf{j}, \mathbf{k}$, and those of the second by $\mathbf{i}', \mathbf{j}', \mathbf{k}'$.   Let

$$(1) \quad \begin{cases} \cos(\mathbf{i}, \mathbf{i}') = \alpha_1, & \cos(\mathbf{i}, \mathbf{j}') = \alpha_2, & \cos(\mathbf{i}, \mathbf{k}') = \alpha_3, \\ \cos(\mathbf{j}, \mathbf{i}') = \beta_1, & \cos(\mathbf{j}, \mathbf{j}') = \beta_2, & \cos(\mathbf{j}, \mathbf{k}') = \beta_3, \\ \cos(\mathbf{k}, \mathbf{i}') = \gamma_1, & \cos(\mathbf{k}, \mathbf{j}') = \gamma_2, & \cos(\mathbf{k}, \mathbf{k}') = \gamma_3. \end{cases}$$

Now, by § 3 eqn. (14),

$$(2) \quad \cos(\mathbf{A}, \mathbf{i}') = \cos(\mathbf{A}, \mathbf{i}) \cos(\mathbf{i}', \mathbf{i}) + \cos(\mathbf{A}, \mathbf{j}) \cos(\mathbf{i}', \mathbf{j}) + \cos(\mathbf{A}, \mathbf{k}) \cos(\mathbf{i}', \mathbf{k}).$$

Let us multiply this equation by A, the magnitude of the vector, remembering that $A \cos(\mathbf{A}, \mathbf{i})$ is equal to A_x, while $A \cos(\mathbf{A}, \mathbf{i}')$ is equal to $A_{x'}$, etc., where A_x and $A_{x'}$ are the x-components in the old and new systems respec-

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[3] In eqns. (17), (18) and (19) the sign is not taken into account.

tively.   We then find for the resulting and the two analogous equations

$$(3) \quad \begin{cases} A_{x'} = \alpha_1 A_x + \beta_1 A_y + \gamma_1 A_z, \\ A_{y'} = \alpha_2 A_x + \beta_2 A_y + \gamma_2 A_z, \\ A_{z'} = \alpha_3 A_x + \beta_3 A_y + \gamma_3 A_z. \end{cases}$$

Conversely, beginning with the second (dashed) system of coordinates, and transforming to the first (undashed) system, we start out from the equation

$$(4) \quad \cos(\mathbf{A}, \mathbf{i}) = \cos(\mathbf{A}, \mathbf{i}') \cos(\mathbf{i}, \mathbf{i}') + \cos(\mathbf{A}, \mathbf{j}') \cos(\mathbf{i}, \mathbf{j}') + \cos(\mathbf{A}, \mathbf{k}') \cos(\mathbf{i}, \mathbf{k}'),$$

and find

$$(5) \quad \begin{cases} A_x = \alpha_1 A_{x'} + \alpha_2 A_{y'} + \alpha_3 A_{z'}, \\ A_y = \beta_1 A_{x'} + \beta_2 A_{y'} + \beta_3 A_{z'}, \\ A_z = \gamma_1 A_{x'} + \gamma_2 A_{y'} + \gamma_3 A_{z'}. \end{cases}$$

The nine *directional cosines*, $\alpha_1, \beta_1 \ldots \gamma_3$, are not independent of each other, but are connected by *six relations* which can readily be derived from the two equations

$$(6) \quad \mathbf{i'i'} = 1, \quad \mathbf{i'j'} = 0.$$

With respect to the first system of coordinates the components of $\mathbf{i}'$ are, according to eqn. (1), $\alpha_1, \beta_1, \gamma_1$; while those of $\mathbf{j}'$ are $\alpha_2, \beta_2, \gamma_2$.   Hence

$$(7) \quad \mathbf{i'i'} = \alpha_1{}^2 + \beta_1{}^2 + \gamma_1{}^2,$$

and

$$(8) \quad \mathbf{i'j'} = \alpha_1\alpha_2 + \beta_1\beta_2 + \gamma_1\gamma_2.$$

By writing down the analogous equations which arise by cyclic interchange, we thus find, in accordance with eqn. (6),

$$(9) \quad \begin{cases} \alpha_1{}^2 + \beta_1{}^2 + \gamma_1{}^2 = 1, \\ \alpha_2{}^2 + \beta_2{}^2 + \gamma_2{}^2 = 1, \\ \alpha_3{}^2 + \beta_3{}^2 + \gamma_3{}^2 = 1, \end{cases}$$

and

$$(10) \quad \begin{cases} \alpha_1\alpha_2 + \beta_1\beta_2 + \gamma_1\gamma_2 = 0, \\ \alpha_2\alpha_3 + \beta_2\beta_3 + \gamma_2\gamma_3 = 0, \\ \alpha_3\alpha_1 + \beta_3\beta_1 + \gamma_3\gamma_1 = 0. \end{cases}$$

Of course, we can also start from the equations

$$(11) \quad \mathbf{ii} = 1, \quad \mathbf{ij} = 0,$$

and replace the scalar products of the fundamental vectors by their components with respect to the second system of

coordinates.   In this way we obtain relations equivalent
to eqns. (9) and (10), *viz.*,

(12)
$$\begin{cases} \alpha_1{}^2 + \alpha_2{}^2 + \alpha_3{}^2 = 1\,, \\ \beta_1{}^2 + \beta_2{}^2 + \beta_3{}^2 = 1\,, \\ \gamma_1{}^2 + \gamma_2{}^2 + \gamma_3{}^2 = 1\,, \end{cases}$$

and

(13)
$$\begin{cases} \alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3 = 0\,, \\ \beta_1\gamma_1 + \beta_2\gamma_2 + \beta_3\gamma_3 = 0\,, \\ \gamma_1\alpha_1 + \gamma_2\alpha_2 + \gamma_3\alpha_3 = 0\,. \end{cases}$$

§ 10. The Gradient of a Scalar Field.

The term *field* signifies a region within which every point
can be associated with a definite value of a certain quantity
which, in general, varies continuously from place to place.
If that quantity is a scalar, we speak of a *scalar field*; if a
vector, of a *vector field*.

Let us consider any point P in a scalar field and construct
a surface, passing through the point, which represents the
geometrical position of all points at which the scalar S has
the same value as at P.   This surface is known as the *field-
level* passing through the point P.   We can then draw a
straight line through P *perpendicular* to the field-level, and
we can construct a *vector* along this line, the magnitude of
which is equal to the *increase of the scalar* in this direction
with respect to the unit of length, the sense being taken as that
of the direction in which the scalar increases in value.   By
this method a vector can be ascribed to every point in the
scalar field, and this vector is called the *gradient of the scalar
field* at the point in question, the symbol *grad S* being used
to denote it.

We will now imagine a coordinate system $(\xi,\ \eta,\ \zeta)$
constructed in such a way that the ξ- and η-axes lie in the
tangent plane drawn at P to the field-level, and that the
direction of the positive ζ-axis is that in which the scalar
increases in value.   Then the following relations hold good
for the components of the gradient :—

(1) $\operatorname{grad}_\xi S = 0\,,\quad \operatorname{grad}_\eta S = 0\,,\quad \operatorname{grad}_\zeta S = \dfrac{dS}{d\zeta}.$

Besides the field-level through P corresponding to a
value S of the scalar, let us suppose a neighbouring level
drawn, which corresponds to the value $(S + dS)$ of the

scalar (Fig. 14). Let $d\zeta$ be the intercept on the ζ-axis between the two field-levels, and let any straight line be drawn from P, which we will call the x-axis. If dx be the intercept on this axis made by the two surfaces,

$$(2) \qquad dx = \frac{d\zeta}{\cos(\zeta, x)},$$

for the ζ-axis is perpendicular to both surfaces, so that a right-angled triangle is formed. Hence the linear rate of increase of the scalar in the direction of x is

$$(3) \qquad \frac{\partial S}{\partial x} = \frac{dS}{d\zeta} \cos(\zeta, x).$$

We will now consider a quite arbitrary system of coordinates (x, y, z), and transform the components of the gradient from the ξ-η-ζ-system into it. Since the ξ- and η-components are zero, we obtain, according to § 9 eqn. (3), the following relations :—

$$(4) \qquad \begin{cases} \operatorname{grad}_x S = \operatorname{grad}_\zeta S \cdot \cos(\zeta, x), \\ \operatorname{grad}_y S = \operatorname{grad}_\zeta S \cdot \cos(\zeta, y), \\ \operatorname{grad}_z S = \operatorname{grad}_\zeta S \cdot \cos(\zeta, z). \end{cases}$$

Hence, by eqn. (3) and the two analogous expressions for $\partial S/\partial y$ and $\partial S/\partial z$, we have, in view of eqn. (1), simply

$$(5) \qquad \begin{cases} \operatorname{grad}_x S = \dfrac{\partial S}{\partial x}, \\ \operatorname{grad}_y S = \dfrac{\partial S}{\partial y}, \\ \operatorname{grad}_z S = \dfrac{\partial S}{\partial z}. \end{cases}$$

The *components of the gradient of a scalar are its partial derivatives with respect to the coordinates.*

If we consider quite generally the field of any vector $\mathbf{A}$, and imagine a curve constructed in the field from a point P_1 to a point P_2, then the integral of the scalar product of the spatially variable vector and the curve-element, taken along the curve, is defined as the *line-integral of the vector along the curve.* It is thus represented by the expression

$$(6) \qquad \int_P^P \mathbf{A}\, d\mathbf{s} = \int_{P_1}^{P_2} (A_x\, dx + A_y\, dy + A_z\, dz).$$

S   S+dS

Fig. 14.
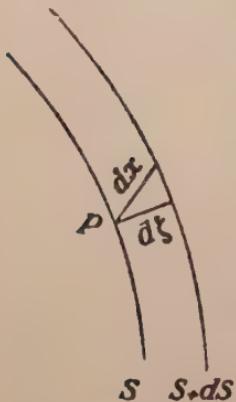
If now the vector **A** happens to be the gradient of a scalar, we have, according to eqn. (5),

$$(7) \qquad \int_{P_1}^{P_2} \operatorname{grad} S \cdot d\mathbf{s} = \int_{P_1}^{P_2} \left(\frac{\partial S}{\partial x} \, dx + \frac{\partial S}{\partial y} \, dy + \frac{\partial S}{\partial z} \, dz \right).$$

But if, as we have assumed to be the case, the value of the scalar at a point is completely determined by the position of the point, *i.e.*, by its coordinates, we then have simply

$$(8) \qquad \frac{\partial S}{\partial x} \, dx + \frac{\partial S}{\partial y} \, dy + \frac{\partial S}{\partial z} \, dz = dS \; .$$

Hence, denoting the values of the scalar S at the points P_1 and P_2 by S_1 and S_2 respectively, we have

$$(9) \qquad \int_{P_1}^{P_2} \operatorname{grad} S \cdot d\mathbf{s} = S_2 - S_1 \; .$$

The *value of the line-integral of the gradient* therefore *depends simply and solely on the difference in value of the scalar* at the two points between which the curve is drawn, the integral being taken along the curve ; on the other hand, it is entirely *independent* of the *form* and *length* of this curve.

If, in particular, the two points P_1 and P_2 are very *close* to each other, and if we denote by $\mathbf{a}_{12}$ the directed line drawn from the first to the second point, we obtain from eqn. (9) a formula which is important for later considerations :

$$(10) \qquad S_2 = S_1 + \mathbf{a}_{12} \cdot \operatorname{grad} S \; .$$

A special case of a scalar field, which is of particular importance in theoretical physics, arises when the scalar forming the field is *only a function of the distance from a fixed point*, so that the field-levels are spherical. If

$$(11) \qquad S = f(r) \, ,$$

we have, for any system of coordinates whose origin coincides with the fixed point,

$$(12) \qquad \operatorname{grad}_x S = f'(r) \frac{\partial r}{\partial x} ,$$

$f'(r)$ being the *derivative* of $f(r)$.   Now

$$(13) \qquad r^2 = x^2 + y^2 + z^2$$

and hence

$$(14) \qquad \frac{\partial r}{\partial x} = \frac{x}{r} \; .$$

Consequently

$$(15) \qquad \operatorname{grad}_x S = f'(r) \frac{x}{r} ,$$

and, quite generally,

$$(16) \qquad \operatorname{grad} f(r) = f'(r)\frac{\mathbf{r}}{r},$$

where $\mathbf{r}/r$ represents the unit-vector which determines the direction of the line joining the fixed point to the point under consideration in the field.

If, in particular, $f(r)$ be equal to r, we shall have

$$(17) \qquad \operatorname{grad} r = \frac{\mathbf{r}}{r},$$

for $f'(r)$ is equal to unity. The gradient of the distance from a fixed point is therefore given by the unit-vector which determines the direction of the line joining the fixed point to the point in question in the field.

If, on the other hand, $f(r)$ be equal to $1/r$, we shall have

$$(18) \qquad \operatorname{grad}\left(\frac{1}{r}\right) = -\frac{1}{r^2}\frac{\mathbf{r}}{r},$$

for $f'(r)$ is equal to $-1/r^2$. The *gradient of the reciprocal distance* from a fixed point is the *reciprocal of the square of the distance*, and its direction is that of the line drawn from the point considered in the field to the fixed point.

§ 11. Potential and Energy.

If a vector can be represented as the gradient of a scalar, the latter, *taken with the opposite sign*, is called the *potential of the vector*. Hence, denoting the potential by Ψ, we have the following simple expression for the line-integral along any given curve :—

$$(1) \qquad \int \mathbf{A}\, d\mathbf{s} = \Psi_1 - \Psi_2,$$

where Ψ_1 and Ψ_2 are the values of the potential for the initial and end-points of the curve. On the other hand, the components of the vector are connected with the potential by the equations

$$(2) \qquad A_x = -\frac{\partial \Psi}{\partial x}, \quad A_y = -\frac{\partial \Psi}{\partial y}, \quad A_z = -\frac{\partial \Psi}{\partial z}.$$

It follows from the definition of potential that, in consequence of the occurrence of arbitrary integration constants, we cannot *measure* potentials absolutely, but only *potential differences*.

The *line-integral of a mechanical force* along a curve is termed the *work* performed by the force along this path :

according as it is positive or negative in sign, we speak of work " *done by* " or " *done against* " the force. If the force is associated with a potential, which is then called a *mechanical potential* and denoted by the letter V, we have

$$(3) \qquad \int \mathbf{K} \, ds = V_1 - V_2 ;$$

and the components of the force are connected with the mechanical potential by the relations

$$(4) \qquad X = -\frac{\partial V}{\partial x}, \quad Y = -\frac{\partial V}{\partial y}, \quad Z = -\frac{\partial V}{\partial z}.$$

Thus, when a force can be derived from a potential, the work is entirely independent of the form and length of the path, being completely determined by the potential difference between the beginning and the end of that path.

It follows from § 10 eqn. (16) that a *force* can always be derived from a *potential* when it is *directed toward a fixed point*, and its magnitude depends *only on its distance* from this point.

Forming the scalar product of the vector equation which expresses Newton's Second Law of Motion [§ 4 eqn. (1)] and the identity

$$d\mathbf{s} = \mathbf{v} \, dt,$$

we find

$$(5) \qquad \mathbf{K} \, d\mathbf{s} = m\mathbf{v} \frac{d\mathbf{v}}{dt} \, dt.$$

Now

$$\mathbf{v} \frac{d\mathbf{v}}{dt} = \frac{1}{2} \frac{d}{dt} (\mathbf{v}\mathbf{v}) = \frac{d}{dt} \left(\frac{v^2}{2} \right);$$

and the product obtained by multiplying half the mass by the square of the velocity is called the *energy of motion* of the moving particle. Denoting it by L, we have

$$(6) \qquad \mathbf{K} \, d\mathbf{s} = \frac{dL}{dt} \, dt,$$

or,

$$(7) \qquad \mathbf{K}\mathbf{v} = \frac{dL}{dt}.$$

If, however, a potential exists, the left-hand side of eqn. (6) is equal to $-dV$, according to eqn. (4): hence

$$-\frac{dV}{dt} = \frac{dL}{dt},$$

whence, on integration,

(8) $L + V$ = constant.

Consequently, if a potential exists, then during the motion
the instantaneous sum of the energy of motion of a moving
particle at any point along its path, and the corresponding
value of the potential associated with that point, must be an
invariable quantity.   In other words, the *sum of the energy
of motion and the mechanical potential* must be a *constant
independent of place and time* : it is known as the *mechanical
energy* of the moving particle.

The conception of energy of motion originated with
Leibniz (1686), who, however, understood by that term
the whole product mv^2.   *Coriolis* was the first to prefix the
factor 1/2 to the product of the mass and the square of the
velocity.   The rudimentary notion of potential can be
traced to *Galileo's* important discovery that the velocity
acquired by a body in sliding down planes, inclined at
various angles, is always the same, provided that the planes
are of the same height.   The real idea of potential is due to
Lagrange and *Laplace*.[1]   Nowadays energy of motion is
universally known as *kinetic energy*, and mechanical potential
as *potential energy*.[2]

§ 12. Rotating Systems of Coordinates.

If a system of coordinates whose fundamental vectors
are i, j, k changes in position relatively to a second system
which we may regard as a standard, then the *fundamental
vectors* themselves become *functions of the time* with respect
to the standard system.   It follows at once from the fact
that the scalar product of a fundamental vector with itself
is equal to unity [§ 3 eqn. (10)] that

(1) $i\dfrac{di}{dt} = 0$, $j\dfrac{dj}{dt} = 0$, $k\dfrac{dk}{dt} = 0$.

Hence the three vectors which represent the time-rates of
change of the fundamental vectors stand *at right angles* to
the corresponding fundamental vectors.[1]

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[1] For the development of the conception of potential, *vide* the author's
"*Grundgleichungen der Mechanik*," lect. 14.   It was not until a paper of
Rankine's was published in 1853 that the word "energy" acquired an extended
use in its modern meaning.

[2] In theoretical physics the unit of work and energy is the *erg*.   It is the
work done by a force of one dyne acting over a length of one centimetre in the
direction of the force.   The work done per unit of time is termed *power*.

[1] According to § 3 eqn. (9).

It can also easily be shown that these three vectors are *co-planar*. Starting with the equation

(2) $$\mathbf{k} = [\mathbf{ij}]$$

[§ 3 eqn. (19)], and differentiating with respect to the time, we have

(3) $$\frac{d\mathbf{k}}{dt} = \left[\frac{d\mathbf{i}}{dt}\mathbf{j}\right] + \left[\mathbf{i}\frac{d\mathbf{j}}{dt}\right].$$

Multiplying vectorially by $d\mathbf{j}/dt$, we find

(4) $$\left[\frac{d\mathbf{j}}{dt}\frac{d\mathbf{k}}{dt}\right] = \left[\frac{d\mathbf{j}}{dt}\left[\frac{d\mathbf{i}}{dt}\mathbf{j}\right]\right] + \left[\frac{d\mathbf{j}}{dt}\left[\mathbf{i}\frac{d\mathbf{j}}{dt}\right]\right].$$

This equation can be transformed in accordance with § 3 eqn. (24),

(5) $$\mathbf{A}\left[[\mathbf{BC}]\right] = \mathbf{B}(\mathbf{CA}) - \mathbf{C}(\mathbf{AB}),$$

and we obtain

(6) $$\left[\frac{d\mathbf{j}}{dt}\frac{d\mathbf{k}}{dt}\right] = \frac{d\mathbf{i}}{dt}\left(\mathbf{j}\frac{d\mathbf{j}}{dt}\right) - \mathbf{j}\left(\frac{d\mathbf{j}}{dt}\frac{d\mathbf{i}}{dt}\right) + \mathbf{i}\left(\frac{d\mathbf{j}}{dt}\frac{d\mathbf{j}}{dt}\right) - \frac{d\mathbf{j}}{dt}\left(\frac{d\mathbf{j}}{dt}\mathbf{i}\right),$$

in which the first term on the right-hand side vanishes, according to eqn. (1). Forming the scalar product of both sides with $d\mathbf{i}/dt$, the third term will then also vanish, by eqn. (1), so that

(7) $$\left\{\begin{aligned}\frac{d\mathbf{i}}{dt}\left[\frac{d\mathbf{j}}{dt}\frac{d\mathbf{k}}{dt}\right] &= -\left(\frac{d\mathbf{i}}{dt}\mathbf{j}\right)\left(\frac{d\mathbf{j}}{dt}\frac{d\mathbf{i}}{dt}\right) - \left(\frac{d\mathbf{i}}{dt}\frac{d\mathbf{j}}{dt}\right)\left(\frac{d\mathbf{j}}{dt}\mathbf{i}\right) \\ &= -\left(\frac{d\mathbf{i}}{dt}\frac{d\mathbf{j}}{dt}\right)\left(\frac{d\mathbf{i}}{dt}\mathbf{j} + \mathbf{i}\frac{d\mathbf{j}}{dt}\right).\end{aligned}\right.$$

Now the scalar product $\mathbf{ij}$ [by § 3 eqn. (11)] vanishes, so that, on differentiating, we find

(8) $$\frac{d\mathbf{i}}{dt}\mathbf{j} + \mathbf{i}\frac{d\mathbf{j}}{dt} = 0.$$

Hence the last term in brackets in eqn. (7) must be equal to zero, and so the left-hand side also vanishes ; whence it follows by § 3 eqn. (22) that

(9) $$\frac{d\mathbf{i}}{dt}, \frac{d\mathbf{j}}{dt}, \frac{d\mathbf{k}}{dt} \text{ are } \textit{co-planar}.$$

If, now, $\mathbf{w_0}$ be a unit-vector which stands at right angles to the plane common to the three vectors $d\mathbf{i}/dt$, $d\mathbf{j}/dt$, $d\mathbf{k}/dt$, then $d\mathbf{i}/dt$ is perpendicular both to $\mathbf{w_0}$ and also, by eqn. (1), to $\mathbf{i}$. We can therefore put $d\mathbf{i}/dt$ equal to the vector product of $\mathbf{w_0}$ and $\mathbf{i}$, multiplied by a scalar (a) which may be positive

or negative, so that the sense of $\mathbf{w}_0$ need not be taken into account for the moment. By forming analogous expressions for $d\mathbf{j}/dt$ and $d\mathbf{k}/dt$, we obtain the relations

(10)　　$\dfrac{d\mathbf{i}}{dt} = a\,[\mathbf{w}_0\,\mathbf{i}]\,, \quad \dfrac{d\mathbf{j}}{dt} = b\,[\mathbf{w}_0\,\mathbf{j}]\,, \quad \dfrac{d\mathbf{k}}{dt} = c\,[\mathbf{w}_0\,\mathbf{k}]\,.$

Substituting these values in eqn. (8), we have

$$\mathbf{j}\,a\,[\mathbf{w}_0\,\mathbf{i}] + \mathbf{i}\,b\,[\mathbf{w}_0\,\mathbf{j}] = 0\,,$$

which, by § 3 eqn. (23), may be written

$$a\,\mathbf{w}_0\,[\mathbf{ij}] + b\,\mathbf{w}_0\,[\mathbf{ji}] = 0\,,$$

or,

(11)　　　　　　$(\mathbf{w}_0\,\mathbf{k})\,(a - b) = 0\,.$

Now it is not possible for the scalar product $\mathbf{w}_0\,\mathbf{k}$ to vanish quite generally, for, since the z-axis is in no way privileged as compared with the x- and y-axes, it follows that the scalar product $\mathbf{w}_0\,\mathbf{k}$ could only vanish in general if $\mathbf{w}_0\,\mathbf{i}$ and $\mathbf{w}_0\,\mathbf{j}$ were also quite generally equal to zero. This, however, is impossible, for, if all three scalar products $\mathbf{w}_0\,\mathbf{i}$, $\mathbf{w}_0\,\mathbf{j}$ and $\mathbf{w}_0\,\mathbf{k}$ vanished together, the unit-vector $\mathbf{w}_0$ would be simultaneously perpendicular to all three coordinate axes, which is naturally out of the question. Consequently, in general, eqn. (11) can only be satisfied if a is equal to b. In like manner we find by a cyclic interchange that b must be equal to c, and c equal to a. The three vectors $d\mathbf{i}/dt$, $d\mathbf{j}/dt$, $d\mathbf{k}/dt$ can therefore be represented as the vector products of *one and the same vector*, which we will call $\mathbf{w}$, and the corresponding fundamental vectors. We may thus put

(12)　　$\dfrac{d\mathbf{i}}{dt} = [\mathbf{wi}]\,, \quad \dfrac{d\mathbf{j}}{dt} = [\mathbf{wj}]\,, \quad \dfrac{d\mathbf{k}}{dt} = [\mathbf{wk}]\,.$

Again, if $\mathbf{A}$ is any vector, we have

(13)　　　　　　$\mathbf{A} = \mathbf{i}\,A_x + \mathbf{j}\,A_y + \mathbf{k}\,A_z\,;$

whence, on differentiating with respect to the time and remembering that the fundamental vectors are not invariable [as was the case in § 3 eqn. (25)], we find

(14)　$\dfrac{d\mathbf{A}}{dt} = \mathbf{i}\dfrac{dA_x}{dt} + \mathbf{j}\dfrac{dA_y}{dt} + \mathbf{k}\dfrac{dA_z}{dt} + A_x\dfrac{d\mathbf{i}}{dt} + A_y\dfrac{d\mathbf{j}}{dt} + A_z\dfrac{d\mathbf{k}}{dt}\,.$

We will now denote the time-rate of change of $\mathbf{A}$ with respect to the coordinate system $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ by the symbol $d^*\mathbf{A}/dt$; *i.e.*,

(15)　　　　　$\dfrac{d^*\mathbf{A}}{dt} = \mathbf{i}\dfrac{dA_x}{dt} + \mathbf{j}\dfrac{dA_y}{dt} + \mathbf{k}\dfrac{dA_z}{dt}\,.$

On the other hand, we find from eqn. (12) that

(16) $A_x \dfrac{d\mathbf{i}}{dt} + A_y \dfrac{d\mathbf{j}}{dt} + A_z \dfrac{d\mathbf{k}}{dt} = [\mathbf{w} \cdot \mathbf{i}\, A_x] + [\mathbf{w} \cdot \mathbf{j}\, A_y] + [\mathbf{w} \cdot \mathbf{k}\, A_z].$

Since, however, the distributive law holds for vector multiplication, the right-hand side of this equation is simply the vector product $[\mathbf{wA}]$, according to eqn. (13). Hence, combining eqn. (14) with eqns. (15) and (16), we obtain the important formula

(17) $$\frac{d\mathbf{A}}{dt} = \frac{d^*\mathbf{A}}{dt} + [\mathbf{wA}].$$

We do not as yet know the signification of the vector $\mathbf{w}$ in this equation. In order to ascertain it, we will consider a directed line ($\mathbf{a}$) drawn from the origin to a *point that is fixed with respect to the coordinate system*, i.e., whose coordinates x, y, z are constant.[2] In this case $d^*\mathbf{a}/dt$ will naturally vanish, and so

(18) $$\frac{d\mathbf{a}}{dt} = [\mathbf{wa}].$$

Let us first consider as a special case that in which the vector $\mathbf{w}$ is *constant both in magnitude and direction* : it will then define a certain direction drawn from the origin, which is shown by OW in Fig. 15. According to eqn. (18), the end of the line $\mathbf{a}$ (represented by OP in the figure) must move in such a way that the element of path described lies at right angles both to the direction OW and to that of OP at the moment under consideration. Thus, if at the instant considered the point P is in the plane of the Fig. 15, the element of path thereupon described by it will be perpendicular to the plane of the figure. Moreover, it will be directed towards the rear, for only then [according to eqn. (18)] will the rotation necessary to carry the direction of $\mathbf{w}$ over into that of $\mathbf{a}$ appear anti-clockwise, as seen from the direction of the path element. If we follow out the movement, we see that it must always

Fig. 15.

be at right angles to OW, and that it also remains at right angles to OP as the direction of the latter changes ; but the distance of P from the origin remains constant. These con-
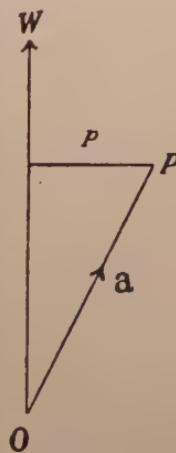
[2] The vector $\mathbf{a}$ has now, of course, an entirely different meaning from that in § **3**.

ditions can only be simultaneously satisfied if the point P describes a *circle in a plane at right angles to the direction* OW. The radius of this circle is equal to the *perpendicular* (p) dropped from the point P on to the direction OW, and the velocity of P is equal to $p \cdot d\varphi/dt$, where $d\varphi$ is the angle through which the perpendicular turns in the interval dt. Hence the magnitude of the left-hand side of eqn. (18) is equal to $p \cdot d\varphi/dt$, while that of the right-hand side is wp, for p is equal to a sin $(\mathbf{w}, \mathbf{a})$. Thus we have the simple relation

$$(19) \qquad\qquad w = \frac{d\varphi}{dt}.$$

Since a by definition and p, in consequence of the circular motion, are both constant, the *angle* included between OP and the direction of the vector must also be *invariable*, provided the vector $\mathbf{w}$ is constant ; that is to say,

(20)   sin $(\mathbf{w}, \mathbf{a})$ = constant, when $\mathbf{w}$ is constant.

The magnitude of the vector $\mathbf{w}$ is hence equal to the angular velocity of rotation of a perpendicular dropped on to the direction of $\mathbf{w}$ from any point fixed with respect to the system of coordinates. The direction and sense of the vector $\mathbf{w}$ are determined by the consideration of the rotation of such a fixed point in a plane at right angles to the direction of $\mathbf{w}$: this rotation must appear anti-clockwise when viewed from the extremity of the vector $\mathbf{w}$ (the motion in Fig. 15 is actually directed to the rear).

The vector $\mathbf{w}$ is called the *angular velocity*. If the origin of a coordinate system does not move, while the axes change their positions, always remaining, however, at right angles to one another, the motion is described as a *turning*, or *rotation*, of the system. If the vector of the angular velocity is not constant, we can only speak of its *instantaneous magnitude* and of its *instantaneous direction*. But even then the magnitude and direction are determined in the afore-mentioned manner, for we can always imagine an element of motion of a point, fixed to a rotating coordinate system, to be replaced by a circular motion.

§ 13. Relative Motion.

Let us consider the motion of a particle, and let us base our considerations upon a certain *standard system*, movement with respect to which we will designate *simply* as *motion*. In contradistinction to this we will describe

as *relative motion* movement with respect to a *second system of coordinates*, whose *fundamental vectors* i, j, k *vary with the time* in regard to the first system.

Let the origin of the first system be O', that of the second being O, and let the moving particle, at a given moment, be at a point P whose coordinates, with respect to the second system, are x, y, z. Let a, r' and r be the directed intervals $O'O, O'P$ and OP respectively (Fig. 16). Then

FIG. 16.

(1) $\mathbf{r'} = \mathbf{a} + \mathbf{r}$.

Introducing now the notation of the previous section, we will define $\mathbf{v}_r$ as the *relative velocity* of the moving particle, where

(2) $$\mathbf{v}_r = \frac{d^*\mathbf{r}}{dt} = \mathbf{i}\frac{dx}{dt} + \mathbf{j}\frac{dy}{dt} + \mathbf{k}\frac{dz}{dt}.$$

Further, we will define as *relative acceleration* the quantity

(3) $$\mathbf{b}_r = \frac{d^*\mathbf{v}_r}{dt};$$

and finally, as the *velocity of translation* of the second system with respect to the first, the quantity

(4) $$\mathbf{v}_t = \frac{d\mathbf{a}}{dt},$$

with the corresponding *acceleration of translation*

(5) $$\mathbf{b}_t = \frac{d\mathbf{v}_t}{dt}.$$

By translation we mean the motion carried out by the *origin* of the second system with respect to the first, so that it follows from what was said in § 12 that the motion of a system of coordinates can always be regarded as the superposition of a rotation upon a translation.

Differentiating eqn. (1) with respect to the time, we have

(6) $$\frac{d\mathbf{r'}}{dt} = \frac{d\mathbf{a}}{dt} + \frac{d\mathbf{r}}{dt}.$$
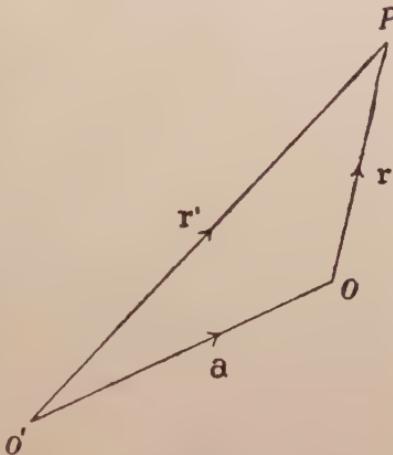
The left-hand side of this equation represents the velocity of the moving particle with respect to the first system of coordinates, *i.e.*, simply the quantity **v**. As regards the right-hand side, the first term gives the velocity of translation, while we find for the second term, in accordance with § 12 eqn. (17),

$$\text{(7)} \qquad \frac{d\mathbf{r}}{dt} = \frac{d^*\mathbf{r}}{dt} + [\mathbf{wr}].$$

Hence, taking into account eqn. (2), eqn. (6) assumes the form

$$\text{(8)} \qquad \mathbf{v} = \mathbf{v}_t + \mathbf{v}_r + [\mathbf{wr}].$$

If we now differentiate with respect to the time, we obtain the acceleration **b** with respect to the first system :

$$\text{(9)} \qquad \mathbf{b} = \mathbf{b}_t + \frac{d\mathbf{v}_r}{dt} + \left[\frac{d\mathbf{w}}{dt}\mathbf{r}\right] + \left[\mathbf{w}\frac{d\mathbf{r}}{dt}\right].$$

Now, by § 12 eqn. (17),

$$\text{(10)} \qquad \frac{d\mathbf{v}_r}{dt} = \frac{d^*\mathbf{v}_r}{dt} + [\mathbf{wv}_r] = \mathbf{b}_r + [\mathbf{wv}_r].$$

Moreover, it follows from eqns. (2) and (7) that

$$\text{(11)} \qquad \left[\mathbf{w}\frac{d\mathbf{r}}{dt}\right] = [\mathbf{wv}_r] + \left[\mathbf{w}\,[\mathbf{wr}]\right].$$

Hence, combining eqns. (10) and (11) with eqn. (9),

$$\text{(12)} \qquad \mathbf{b} = \mathbf{b}_t + \mathbf{b}_r + 2[\mathbf{wv}_r] + \left[\frac{d\mathbf{w}}{dt}\mathbf{r}\right] + \left[\mathbf{w}\,[\mathbf{wr}]\right].$$

Assuming for the moment that the particle is rigidly connected with the coordinate system (**i, j, k**), the values of its velocity and acceleration then represent the so-called *velocity* and *acceleration of* " *transport* " ("*Führungsgeschwindigkeit*," *etc.*). We will denote these quantities by $\mathbf{v}_f$ and $\mathbf{b}_f$: their values can be obtained from eqns. (8) and (12) by putting the relative velocity and relative acceleration equal to zero. Hence we have

$$\text{(13)} \qquad \mathbf{v}_f = \mathbf{v}_t + [\mathbf{wr}]$$

and

$$\text{(14)} \qquad \mathbf{b}_f = \mathbf{b}_t + \left[\frac{d\mathbf{w}}{dt}\mathbf{r}\right] + \left[\mathbf{w}\,[\mathbf{wr}]\right].$$

Consequently,

$$\text{(15)} \qquad \mathbf{v} = \mathbf{v}_f + \mathbf{v}_r,$$

and

$$\text{16)} \qquad \mathbf{b} = \mathbf{b}_f + \mathbf{b}_r + 2\,[\mathbf{wv}_r].$$

Thus, whereas the velocity is equal to the vectorial sum of the relative and transport velocities, an analogous relation does not hold in the case of the acceleration. In order to obtain the latter, we have to add to the vectorial sum of the relative and transport accelerations twice the vector product of the angular and relative velocities. This double vector product is called the *Coriolis acceleration*, after the physicist Coriolis, to whom we owe the most important theorems on the subject of relative motion (1829).

On multiplying eqn. (16) by the mass of the moving particle, we obtain on the left-hand side the force. If we wish to retain Newton's Second Law of Motion for relative motion also, in virtue of which law the product of the mass and the relative acceleration should be equal to the force, we should have to add *two imaginary accessory forces equal and opposite* to the products of the mass and the transport acceleration, or the Coriolis acceleration respectively. These accessory forces are hence called the *force of transport* and the *Coriolis force*.

We will now transform the double vector product which occurs in eqn. (12). According to § 3 eqn. (24),

$$(17) \quad \Big[\mathbf{w} \, [\mathbf{wr}] \Big] = \mathbf{w} \, (\mathbf{rw}) - \mathbf{r} \, (\mathbf{ww}) = \mathbf{w} \, rw \cos (\mathbf{r}, \mathbf{w}) - \mathbf{r} w^2 \, .$$

Let $\mathbf{w_0}$ be the unit-vector corresponding to the direction of the vector $\mathbf{w}$, and let r_w be the projection of $\mathbf{r}$ upon the direction of $\mathbf{w}$. Substituting these in eqn. (17), we have

$$(18) \qquad \Big[\mathbf{w} \, [\mathbf{wr}] \Big] = w^2 \, (\mathbf{w_0} \, r_w - \mathbf{r}) \, .$$

Now, drawing the directions of $\mathbf{w}$ and $\mathbf{r}$ from a point O, and dropping a perpendicular $\mathbf{p}$ from the position of the particle on to the direction of $\mathbf{w}$, we see from Fig. 17 that

$$(19) \qquad \mathbf{w_0} \, r_w = \mathbf{r} + \mathbf{p} \, .$$

Hence eqn. (18) becomes

$$(20) \qquad \Big[\mathbf{w} \, [\mathbf{wr}] \Big] = \mathbf{p} \, w^2 \, .$$

If, in particular, the vector of angular velocity is *constant* both in magnitude and direction, and there is no translational acceleration, it follows from eqn. (14) that the expression $\mathbf{p} \, w^2$ represents the transport acceleration. The force of transport, which

FIG. 17.

is equal and opposite to the product of the mass and the transport acceleration, is then given by

$$(21) \qquad K_f = - m \, \mathbf{p} \, w^2 \, .$$
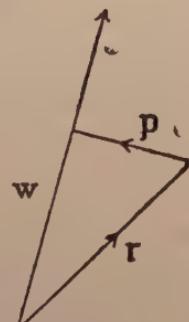
Hence, in the case under consideration, the force of transport, whose sign is opposite to that of the vector **p**, reveals itself in the form of a *centrifugal force*. Its magnitude is equal to the product of the mass, the square of the angular velocity, and the perpendicular dropped from the moving particle on a line which, passing through the origin, represents the direction of the angular velocity.

We may finally discuss eqn. (12) for the special case that the vector **w** vanishes, so that the axes of the second system remain *constantly parallel* to those of the first. We then have

$$(22) \qquad\qquad \mathbf{b} = \mathbf{b}_t + \mathbf{b}_r.$$

Limiting our conditions still further to the instance that the *motion of the origin* of the second system with respect to the first is *uniform,* we have simply $\mathbf{b}_r$ equal to $\mathbf{b}$: the acceleration is then identical for both systems. Hence *two coordinate systems in a state of uniform translatory motion with respect to each other are completely equivalent* for the description of mechanical processes. This important conclusion is known as the *mechanical principle of relativity.*

In general, however, the transition from one coordinate system to a second, which is moving in any arbitrary way with respect to the first system, demands the introduction of accessory forces. It follows from this that there must be a certain privileged class of coordinate systems which are all in a state of uniform motion with respect to each other, and which are distinguished by the fact that *accessory forces vanish* for all of them. It is only with respect to such systems that a particle, which is not acted upon by any external forces and which is moving only under the influence of its inertia, maintains its velocity unchanged both in magnitude and direction. These privileged coordinate systems are hence known as *inertial systems*, or also as *mechanical fundamental systems* : experience shows that such an inertial system is determined by the *firmament of fixed stars.*

§ 14. Processes of Motion on the Rotating Earth.

Far and away the most important *relative motions* in physics are those which occur *on the rotating earth.* The direction of the vector of the *angular velocity* of the earth, and of every coordinate system rigidly connected to it, is determined by the *earth's axis* which runs from one Pole to the other.[1] Since the rotation of the earth, from West to

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[1] The motion of the earth, apart from its rotation, can be regarded as uniform and linear, and so left out of account.

East, appears anti-clockwise when seen from the North Pole, the *sense* of the vector of angular velocity is directed *from the South to the North Pole*. Moreover, the *magnitude* of the angular velocity measured in radians is equal to 2π divided by the number of seconds in a day : hence

(1) $w = 7 \cdot 272 \times 10^{-5}$ sec. [1]

Let us now consider a point on the earth's surface and suppose it to be the origin of a system of coordinates rigidly connected to the earth : we can then represent the constant angular velocity by a straight line drawn from this point and making constant angles with the coordinate axes [according to § 12 eqn. (20)]. Since this directed line is parallel to the earth's axis, it lies in the *meridian plane*, and makes an angle with the horizontal equal to the *geographical latitude* of the place in question. In the *northern hemisphere* it therefore points *upwards*, and in the *southern*, *downwards* or towards the earth. At the Equator it is exactly horizontal.

Since the transport acceleration is proportional to w^2, whereas the Coriolis acceleration is proportional to wv, the former acceleration *can be neglected* in comparison with the latter, because of the smallness of w, provided that the relative velocity is not too small, and that the motion does not lead too far away from the point in question on the earth's surface. If, for example, the distance from the origin is not greater than about 200 metres, the transport acceleration will not be greater than about 10^{-4} cm. sec.$^{-2}$ On the other hand, the magnitude of the Coriolis acceleration is of the order 10^{-2} cm. sec.$^{-2}$, *i.e.*, roughly a hundred times as great, when the velocity is about one metre per second.

Since the Coriolis force is equal and opposite to the product of the mass and the Coriolis acceleration, and since a vector product changes in sign on interchanging the factors, we have for the Coriolis force [by § 13 eqn. (16)]

(2) $\mathbf{K}' = 2m \ [\mathbf{v_r w}]$,

where m is the mass of the moving body.

As a first example of motion on the rotating earth, we will take the case of a *freely-falling body*. The vector $\mathbf{v_r}$ is then directed vertically downwards, and the vector $\mathbf{w}$ points obliquely from South to North. By the definition of a vector product, and by eqn. (2), the sense of $\mathbf{K}'$ must be such that the rotation which would carry $\mathbf{v_r}$ over to the direction of $\mathbf{w}$ appears anti-clockwise, when seen from the end of $\mathbf{K}'$. Hence, both in Fig. 18, which refers to the northern hemisphere, and in Fig. 19, which refers to the southern,

the vector **K**′ must be directed to the rear : the Coriolis force always produces an *easterly deviation* in a freely-falling body.

Since the vector **w** makes an angle with the vertical
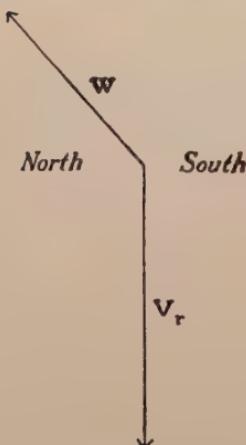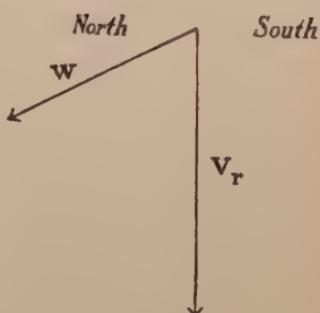
which is the complement of the geographical latitude (ψ), the value of the Coriolis force, according to eqn. (2), is given by

$$(3) \qquad K' = 2m\,v_r w \cos \psi.$$

Now, the relative velocity is equal to gt, where g is the acceleration due to gravity. Hence, if x is the easterly deviation,

$$(4) \qquad \frac{d^2x}{dt^2} = 2gtw \cos \psi;$$

whence, by integrating twice in succession,

$$(5) \qquad x = \frac{gwt^3}{3} \cos \psi.$$

(The integration constants can be omitted, since at the time $t = 0$ both x and $dx/dt = 0$.) Since the height of fall is given by

$$h = \frac{gt^2}{2},$$

the total easterly deviation can also be expressed by the formula

$$(6) \qquad x = \frac{2\sqrt{2}}{3} \frac{h^{\frac{3}{2}}}{\sqrt{g}} w \cos \psi.$$

The deviation is greatest at the Equator, while it vanishes entirely at the Poles. If we take London as an example, the latitude is $51\frac{1}{2}°$, and we have

(7) $$w \cos \psi = 4 \cdot 527 \times 10^{-5} \text{ sec.}^{-1}$$

Since g is equal to 981 cm. sec.$^{-2}$, it follows from eqn. (6) that, for a height of fall of 100 metres in the latitude of London, the easterly deviation amounts to 1·36 cm.

We will take as a second example of relative motion on the rotating earth, the case of *horizontal motion*. The direction of the Coriolis force must then be such that, looking against its line of action, the rotation which would carry the direction of the horizontal motion round to that of the vector representing the earth's angular velocity appears anti-clockwise. As was explained above, however, this vector is always directed upwards in the northern hemisphere, and downwards in the southern. Now an upward movement from a horizontally directed line appears anti-clockwise to a spectator when he is situated on the right-hand side of that line. Hence the tendency of the Coriolis force is to *deflect every horizontal motion, in the northern hemisphere towards the right*, and *in the southern hemisphere towards the left*.

In consequence of this lateral deflection, currents in the sea and in the atmosphere deviate towards the right in the northern hemisphere, and towards the left in the southern. Consequently, a north wind in the northern hemisphere tends gradually to change into an east wind ; for an East-West direction is equivalent to the direction left to right for a man walking from North to South.[2]

In order to gain a definite idea as to the magnitude of the lateral deviation in horizontal motion, we shall consider the simple case in which the horizontal motion begins in the meridian plane. The vector **w** then makes an angle with the direction of the relative velocity equal to the geographical latitude ψ ; and so, according to eqn. (2), the second derivative of the deviation with respect to the time is given by

(8) $$\frac{d^2x}{dt^2} = 2vw \sin \psi.$$

[2] This fact gives the explanation of the *trade-winds*. Air expanded by heat in the tropics rises vertically upwards and flows towards either Pole, forming a strong current. At a lower level, masses of colder air consequently flow north and south towards the Equator, thereby completing the circulation. Owing to the lateral deviation, these lower currents appear as north-east trade-winds in the northern hemisphere, and south-east trade-winds in the southern. In the upper regions of the air, the currents blowing from the Equator to the Poles form the anti-trade winds. In the northern hemisphere this wind is south-west ; in the southern, north-west.

Differentiating twice in succession, we find

(9) $x = vt^2w \sin \psi.$

In the latitude of London $w \sin \psi$ is equal to $5 \cdot 691 \times 10^{-5}$ sec.$^{-1}$ Hence, if a missile has an initial velocity of 500 metres a second, the lateral deviation after one second amounts to $2 \cdot 7$ cm. ; after two seconds it is four times as great, and so on.

In selecting a third important example of a process of motion on the rotating earth, we will finally consider a *system of coordinates* which is only *partially connected with the earth*. We will suppose that the z-axis is kept constantly *vertical*, but that the x- and y-axes only partake in the earth's rotation in so far as they must always remain at right angles to the vertical z-axis ; apart from this, they are naturally quite free to move in a horizontal plane.

Let us next imagine two coordinate systems at any point on the earth, one of which represents an inertial system, while the other is rigidly connected to the earth. Then the second system rotates with an angular velocity **w** with respect to the first, while, conversely, the inertial system appears to an observer on the earth to rotate with an angular velocity $(-\mathbf{w})$. Now we can resolve the angular velocity **w** into three components with respect to the coordinate system rigidly connected to the earth ; and, if we so arrange this system that the z-axis is vertical and the horizontal x-axis has the direction of the meridian, we have :

(10) $w_x = w \cos \psi, \quad w_y = 0, \quad w_z = w \sin \psi.$

According to the definition of angular velocity, w_x here signifies the angular velocity about the x-axis (*i.e.*, in the y-z-plane), w_y that about the y-axis, and w_z the angular velocity about the vertical axis in a horizontal plane.

A system of coordinates in which only the z-axis is always vertical and rigidly connected to the earth, while the other two axes can turn in a horizontal plane, will consequently appear to an observer on the earth to rotate with an angular velocity opposite and equal to w_z, and given by

(11) $w' = -w \sin \psi.$

We will now consider a *pendulum* which is set in motion on the rotating earth, without any initial lateral velocity ; the pendulum is displaced from its position of rest, and it is then released without any impulse being imparted to it. Since the position of rest must always be vertically beneath

the point of suspension, in consequence of the earth's gravity, the pendulum preserves its direction of oscillation with respect to a coordinate system which is partially connected to the earth in the manner described above; *viz.*, a system in which one axis is always vertical, while the other two can rotate in a horizontal plane.   Hence it follows from what has been previously stated that, *to an observer on the earth*, the *vertical plane of oscillation* of the pendulum will appear to *rotate* :   according to eqn. (11) it will rotate in one day through an angle of 360° multiplied by the sine of the geographical latitude.   On account of the negative sign in eqn. (11), the direction of the rotation in the northern hemisphere will be such that, when seen *from above*, it appears *clockwise*.   It was in this way that *Foucault* was able, in 1850, to establish a direct *experimental proof of the earth's rotation*, by means of his famous pendulum experiment.[3]

[3] In Foucault's pendulum experiment, the most essential point is the complete avoidance of an initial lateral velocity, for the latter will also produce a rotation of the plane of oscillation.   In such an experiment, therefore, the pendulum is pulled aside from its position of rest by means of a cord, and held there motionless :   the cord is then burned through.

Chapter II

THE GENERAL PRINCIPLES OF MECHANICS

§ 15. The Law of the Centre of Mass.

HITHERTO we have been considering the dynamics of an *individual* particle, but we shall now proceed to *systems* of any number of particles which are acted upon by arbitrary external and internal forces, without, however, being in any way limited as regards their freedom of movement. Such forces as act *between* the individual particles will be designated *internal forces*, while all others will be termed *external*.

We will suppose the total number of particles which make up the system to be n, and that they are numbered consecutively from 1 to n; and we will begin by considering an arbitrary individual particle, number h. Let its mass be m_h; its momentum $\mathbf{G}_h$; the resultant of all the external forces acting on it $\mathbf{K}_h$; and the internal force exerted upon it by the particle k, $\mathbf{K}'_{hk}$. Then the *equation of motion of the hth particle*, in vectorial notation, will be

$$(1) \qquad \frac{d\mathbf{G}_h}{dt} = \mathbf{K}_h + \sum_{k=1}^{k=n}{}' \mathbf{K}'_{hk}.$$

The dashed summation symbol indicates that the sum is to be taken over all integral values of k from 1 to n, *with the exception of h itself*.

Therefore, whatever the composition of the system may be, and whatever the nature of the forces which act on it, in every case $3n$ equations of motion can be analytically written down, which connect the second derivatives of the coordinates of the particles with the forces acting on the system, *e.g.*,

$$m_h \frac{d^2 y_h}{dt^2} = Y_h + \sum_{k=1}^{k=n}{}' Y'_{hk}.$$

Where, now, forces are acting between the individual particles, which depend on the mutual *distances* between them, these *forces* will themselves be *functions of the coordinates*. Consequently, although the *formation of the $3n$ differential equations of the second order* is in itself always possible, as soon as the

position of the system is to be determined for a given moment, colossal *mathematical difficulties* in general arise; for, in order to do this, the $3n$ differential equations must be *integrated twice*, and that is usually such a difficult problem that, analytically, it has not yet been completely mastered.

By means, however, of the *simple methods of integration* which have already been applied in the dynamics of the individual particle, *seven first integrals* of the $3n$ differential equations of the second order can be obtained. These lead to the so-called *Three Integral Laws of Mechanics*, which render superfluous the carrying out of the integration in every special case; for, by insertion of the special values in them, the required first integrals are at once obtained for the special case in question.

The *first* integral law is based on *Newton's Third Law of Motion*, the so-called principle of the *equality of action and reaction*.[1] In accordance with this principle the force with which the kth particle acts upon the hth must be equal and opposite to that with which the hth particle acts upon the kth, so that we have

(2) $$\mathbf{K}'_{hk} + \mathbf{K}'_{kh} = 0.$$

In this equation h is first put $= 1$, and the equation is then formed for all values of k, except for $k = 1$. Next let h be put $= 2$, and the equation again formed for all values of k, except for $k = 2$. This process is continued until, finally, h is put $= n$, and the equation then formed for all values of k except that of n. In this way $n\,(n-1)$ equations are obtained, but it will be seen that every force occurs *twice* in this system; for, in the $n\,(n-1)$ equations, there occur, for example, both the equations

$$\mathbf{K}'_{4,7} + \mathbf{K}'_{7,4} = 0$$

and

$$\mathbf{K}'_{7,4} + \mathbf{K}'_{4,7} = 0 .$$

Hence, if all the $n\,(n-1)$ equations are added together and divided by 2, we have

(3) $$\sum_{h=1}^{h=n} \sum_{k=1}^{k=n}{}' \mathbf{K}'_{hk} = 0 .$$

From this formula, if eqn. (1) be formed for all values of h

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[1] *Lex* III. : " Actioni contrariam semper et æqualem esse reactionem ; sive corporum duorum actiones in se mutuo semper esse æquales et in partes contrarias dirigi."

from 1 to n, and these n equations be then added up, we obtain the important relation

(4) $$\frac{d}{dt}\left(\sum \mathbf{G}_h\right) = \sum \mathbf{K}_h.$$

We see from this equation that the *total momentum of a system*, upon which *no external forces* act, must remain unchanged—a law that was first formulated by *Descartes* [2] in its application to the universe, though without any exact proof. Moreover, the connection which exists between momentum and force in the case of an individual particle [§ 4 eqn. (15)] appears in an *extended* form in eqn. (4), which applies to a system of masses.

Eqn. (4) leads to a particularly simple law, when use is made of an idea, by the introduction of which *Archimedes* really opened up the development of theoretical physics. That is the idea of the *centre of mass*. The mass-centre of two masses is defined as a point which divides the line joining them inversely as their masses : thus, if we call S the centre of mass of two masses located at the points A and B, we have the ratio

$$AS : SB = m_2 : m_1,$$

and hence,

(5) $$AS = \frac{m_2 \cdot AB}{m_1 + m_2}.$$

From this follows at once the analytical expression for the position of the centre of mass of two masses m_1 and m_2 which are placed at the points (x_1, y_1, z_1) and (x_2, y_2, z_2). For, if x', y', z' are the coordinates of the centre of mass, we have, according to eqn. (5), as a simple geometrical consideration shows,

$$x' - x_1 = \frac{m_2(x_2 - x_1)}{m_1 + m_2},$$

or,

(6) $$x' = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2};$$

with two analogous equations for y' and z'.

If it is a question of determining the centre of mass of *three* masses, *e.g.*, the two above-mentioned and a third mass (m_3) at the point (x_3, y_3, z_3), this mass-centre will have to be *defined* as that of a mass $(m_1 + m_2)$, which we may imagine concentrated at the point (x', y', z'), and of the third

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[2] *Cf.* the author's "*Grundgleichungen der Mechanik*," lect. 6.

mass m_3. We find by eqn. (6) that the x-coordinate of the centre of mass of the three masses is given by

$$x'' = \frac{(m_1 + m_2)x' + m_3 x_3}{(m_1 + m_2) + m_3} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3}{m_1 + m_2 + m_3}.$$

By extending this definition to a system of any number of particles, we obtain for the mass-centre of the whole system, whose position may be represented by the coordinates ξ, η, ζ, the analytical expression

(7)
$$\begin{cases} \xi = \dfrac{\sum m_h x_h}{\sum m_h}, \\[2mm] \eta = \dfrac{\sum m_h y_h}{\sum m_h}, \\[2mm] \zeta = \dfrac{\sum m_h z_h}{\sum m_h}. \end{cases}$$

Differentiating eqn. (7) with respect to time, we have (omitting the index h for the sake of simplicity),

$$\frac{d\xi}{dt} \sum m = \sum G_x;$$

and consequently, on further differentiation, by again changing the order of the summation and differentiation symbols,

$$\frac{d^2 \xi}{dt^2} \sum m = \sum \frac{dG_x}{dt}.$$

Thus, by introducing the idea of centre of mass, eqn. (4) assumes the analytical form

(8)
$$\begin{cases} \dfrac{d^2 \xi}{dt^2} \sum m_h = \sum X_h, \\[2mm] \dfrac{d^2 \eta}{dt^2} \sum m_h = \sum Y_h, \\[2mm] \dfrac{d^2 \zeta}{dt^2} \sum m_h = \sum Z_h. \end{cases}$$

These equations can be expressed in the form that the *centre of mass* of a system, on which *only internal forces* are acting, must either be *at rest* or proceeding with *linear, uniform motion* ; that, on the other hand, if external forces are acting on the system, the latter behaves as though its *whole mass were concentrated at the centre of mass*, and as though all the external forces (which in reality act on the individual particles) were operating with the *same intensity*

and in the *same direction* on the mass so *concentrated* at this centre.

The result derived above for a system of free particles maintains its validity also for solid bodies, as will be shown in § 23. Since the *gravitational forces* acting on the individual particles of a solid body are *parallel*, the total mass of the body can accordingly be regarded as concentrated at its centre of mass, and its whole weight as operating at this point. It can hence be seen why the centre of mass of a system is also generally termed its *centre of gravity*; and, since in the absence of external forces the centre of gravity, if once at rest, must *always* remain at rest [according to eqn. (8)], the first integral law of mechanics, treated in this section, is also known in brief as the Principle of the *Conservation of the Centre of Gravity*.

§ 16. The Law of the Conservation of the Total Angular Momentum.

Just as the principle of inertia, which holds for an individual particle, was generalized in the previous section, so also the *law of areas* deduced in § 7 can be *extended* to a system of any number of particles, provided certain hypotheses have validity. If § 15 eqn. (1) be vectorially multiplied with the radius-vector, denoted by $\mathbf{r}$, which is drawn from a fixed reference point, we obtain

$$(1) \qquad \left[\mathbf{r}_h \cdot \frac{d\mathbf{G}_h}{dt} \right] = [\mathbf{r}_h \cdot \mathbf{K}_h] + \sum_{k=1}^{k=n}{}' [\mathbf{r}_h \cdot \mathbf{K}'_{hk}].$$

If we call the connecting line drawn from the hth to the kth particle $\mathbf{r}_{hk}$, so that

$$\mathbf{r}_{hk} = \mathbf{r}_k - \mathbf{r}_h,$$

we find, by making use of § 15 eqn. (2), that

$$(2) \qquad [\mathbf{r}_h \cdot \mathbf{K}'_{hk}] + [\mathbf{r}_k \cdot \mathbf{K}'_{kh}] = - [\mathbf{r}_{hk} \cdot \mathbf{K}'_{hk}].$$

The vector product on the right-hand side of this equation, however, vanishes [according to § 3 eqn. (17)] when the two vectors multiplied together are *similarly directed*; or, in other words, when the *forces* operating between the individual particles act *along the lines joining them*, i.e., when they are so-called *central forces*.

Therefore, on forming eqn. (1) for all the particles and adding together the n equations so obtained, the double sum on the right-hand side vanishes, according to eqn. (2).[1]

[1] The method of reasoning leading to this is exactly the same as that carried out in § 15.

Hence, on first multiplying eqn. (1) by the mass of the particle in question, and then adding, we obtain the simple relation [2]

$$(3) \qquad \frac{d}{dt}\left(\sum \mathbf{U}_h\right) = \sum \mathbf{M}_h\,;$$

where $\mathbf{U}_h$ is the angular momentum of the hth particle, and $\mathbf{M}_h$ the moment of the resultant of all the external forces acting upon it.

The total angular momentum of a system therefore remains unchanged in magnitude and direction, provided either that only internal forces are acting along the lines joining the masses, or that, apart from these, only such external forces are acting as are directed towards the point to which the moments are referred.[3] This law is known as the Law of the *Conservation of the Total Angular Momentum*.[4]

Naturally both the size and direction of the total angular momentum in general depend on the position of the reference point. Let the total angular momentum referred to a certain point P be $\mathbf{U}$, and suppose that we wish to calculate its amount $\mathbf{U}'$ with reference to another point P', the directed interval PP' being denoted by $\mathbf{d}$. If $\mathbf{r}$ is the radius-vector drawn to a particle of the system from P, and $\mathbf{r}'$ that drawn to the same particle from P', we have

$$(4) \qquad \mathbf{r}' = \mathbf{r} - \mathbf{d}\,,$$

and hence

$$(5) \qquad \sum [\mathbf{r}'\mathbf{G}] = \sum [\mathbf{r}\mathbf{G}] - [\mathbf{d}\,.\,\sum \mathbf{G}],$$

or,

$$(6) \qquad \mathbf{U}' = \mathbf{U} - [\mathbf{d}\,.\,\sum \mathbf{G}].$$

As is obvious from this equation, both the magnitude and direction of the total angular momentum are *completely independent* of the position of the point to which the moments are referred, when $\sum \mathbf{G}$ is equal to zero, or, what is exactly

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[2] According to § 7 eqn. (7).

[3] For, in that case, the right-hand side of eqn. (3) also vanishes.

[4] A series of very striking examples illustrating this principle is to be found in Mach's "*Die Mechanik in ihrer Entwicklung*," and in Föppl's "*Vorlesungen über technische Mechanik*," vol. 4, § 14. We shall only mention one here. If only such railway trains and ships were travelling on the earth as moved parallel to the Equator from West to East, the rotation of the earth (likewise from West to East) would be thereby retarded and the day lengthened. At the moment that all these trains and ships came to a standstill, the day would regain its normal length. Were they now all to reverse their direction and move from East to West, the day would become shorter.

the same thing (according to § 15), when the *centre of gravity* of the system is *at rest*.[5]   Moreover, in the case of progressive motion of the centre of gravity, the angular momentum (referred, for instance, to that centre) must, according to eqn. (3), represent a vector independent of the time, provided the system is subjected only to internal central forces. Consequently, in every such system a *plane* which passes through the centre of gravity, and which is perpendicular to the vector of the angular momentum, must maintain its *position unaltered*.   Hence there must be an *invariable plane* in the *solar system* also, as was first recognized by *Laplace*.

§ 17. The Law of the Conservation of Mechanical Energy.

Besides the *integration methods* which lead to the Laws of the Conservation of the Centre of Gravity and the Conservation of Angular Momentum, a third integration method is available for the equations of motion of a system of particles, if we take certain hypotheses for granted : this method, in its application to the *individual* particle in § 11, gave us the Law of the *Conservation of Mechanical Energy*.

We begin again with the equation of motion of the hth particle [§ 15 eqn. (1)],

$$(1) \qquad m_h \cdot \frac{d\mathbf{v}_h}{dt} = \mathbf{K}_h + \sum_{k=1}^{k=n}{}' \mathbf{K}'_{hk} \,.$$

If we form the scalar product of this equation and the identity

$$\mathbf{v}_h \cdot dt = d\mathbf{s}_h$$

(in which $d\mathbf{s}_h$ represents the element of path traversed by the hth particle in the interval dt), we find

$$(2) \qquad m_h \cdot d\left(\frac{v_h{}^2}{2}\right) = \mathbf{K}_h \cdot d\mathbf{s}_h + \sum_{k=1}^{k=n}{}' \mathbf{K}'_{hk} \cdot d\mathbf{s}_h \,.$$

We will now consider the especially simple case that not only the external forces $\mathbf{K}_h$ are derived from potentials V_h, but also that there exists a function of the coordinates V', which may be termed the *internal potential* of the system, and which is of such a nature that its negative partial derivatives, with respect to the coordinates of any particle,

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[5] For the momentum of a system is equal to the momentum of its mass supposed concentrated at the centre of gravity. Moreover, as appears from eqn. (6), $\mathbf{U}'$ is also equal to $\mathbf{U}$ in the special case in which the line joining the two reference points has the same direction as the vector representing the total momentum.

give the components of the total internal force acting on that particle: we then have

$$-\frac{\partial V'}{\partial x_h} = \sum_{k=1}^{k=n} {}' X'_{hk} , \; etc.$$

In virtue of the considerations of § 10, it will be seen that such an internal potential will always exist, when the *internal forces* are *central* and *only functions of the mutual distances between the particles.* For, if the force is only a function of the mutual distance (and otherwise only dependent on constant quantities), we can put

$$(3) \qquad K'_{hk} = \varphi(r_{hk}) = \frac{d\Phi(r_{hk})}{dr_{hk}};$$

and, if the force is central, we have, further,

$$(4) \qquad X'_{hk} = K'_{hk} \cdot \frac{x_k - x_h}{r_{hk}}.$$

Hence we have, in point of fact, by § 10 eqn. (14),

$$(5) \qquad \frac{\partial \Phi}{\partial x_h} = \frac{d\Phi}{dr_{hk}} \cdot \frac{\partial r_{hk}}{\partial x_h} = \varphi \cdot \frac{x_k - x_h}{r_{hk}} = - X'_{hk} .$$

If we now form the double sum

$$\sum_{h=1}^{h=n} \sum_{k=1}^{k=n} {}' \Phi(r_{hk}),$$

partial differentiation with respect to the x-coordinate of any arbitrary particle (number i) gives the value

$$- 2 \sum_{p=1}^{p=n} {}' X'_{ip} ;$$

for the partial derivative with respect to x_i always vanishes, if neither h nor k be equal to i. On the other hand, the value X'_{ip} occurs *twice*; namely, once when $h = i$ and $k = p$, and secondly when $h = p$ and $k = i$. Hence the internal potential of the system is given by:

$$(6) \qquad V' = \frac{1}{2} \sum_{h=1}^{h=n} \sum_{k=1}^{k=n} {}' \Phi(r_{hk}).$$

If, in particular, the internal forces obey *Newton's Law of Gravitation*, so that [by § 8 eqn. (19)]

$$\varphi(r_{hk}) = \kappa \frac{m_h m_k}{r_{hk}{}^2}, \quad \Phi(r_{hk}) = -\kappa \frac{m_h m_k}{r_{hk}},$$

the expression becomes

$$(7) \qquad V' = -\frac{1}{2} \sum_{h=1}^{h=n} \sum_{k=1}^{k=n} {}' \kappa \frac{m_h m_k}{r_{hk}} .$$

If we now denote the *kinetic energy* of an individual particle by L, addition of the n equations (2)—provided that, besides the existence of an internal potential, the external forces may also be derived from potentials V_h—gives the formula

$$(8) \qquad d\,(\Sigma L_h) = -\,d\,(\Sigma V_h) - dV'.$$

The sum ΣL_h, which represents the total kinetic energy of the system, may now be denoted by L : we shall call ΣV_h the external potential of the system (V_a) and, finally, the sum of the external and internal potential we shall term the total potential (V), or the total potential energy of the system. Integration of eqn. (8) consequently leads to the formula

$$(9) \qquad\qquad L + V = W,$$

where W is an integration constant. Hence, in a system of masses, exactly as in the case of an individual particle, there exists in the presence of a potential the important relation that the sum of the kinetic and potential energies represents a quantity that is independent of time and place, and which is *invariable* during the whole motion ; this quantity is called the *mechanical energy of the system.*

Since the value of the internal potential only depends on the distances between the individual particles of the system (*i.e.*, only on the values of r_{hk}), it follows from eqn. (9) that the *kinetic energy* of a system which is not subjected to external forces, and within which only central forces are acting, must always assume the *same value*, when the system again takes up the *same configuration.* The system therefore *preserves* a definite value of kinetic energy for a definite configuration ; and, in consequence of this fact, discovered by *Leibniz* and termed by him " *conservatio virium vivarum*," such a system may be called a *conservative* system. In general, all forces which can be derived from a potential are called *conservative forces.*

Like potential, *kinetic energy* can also be resolved into two components, which may be distinguished as *internal* and *external* kinetic energy. If we call the coordinates of the system *relative to the centre of gravity* x', y', z', and those of the centre of gravity itself x_0, y_0, z_0, we have

$$(10) \qquad\qquad x = x' + x_0,$$

and hence

$$(11) \quad \begin{cases} \dfrac{1}{2}\sum m \left(\dfrac{dx}{dt}\right)^2 = \dfrac{1}{2}\sum m \left(\dfrac{dx'}{dt}\right)^2 + \dfrac{dx_0}{dt}\cdot\dfrac{d}{dt}\left(\sum mx'\right) \\ \qquad\qquad + \dfrac{1}{2}\left(\dfrac{dx_0}{dt}\right)^2 \sum m\,. \end{cases}$$

Now, according to § 15 eqn. (7), $\Sigma\, mx'$ is equal to the total mass multiplied by the x'-coordinate of the centre of gravity itself, and this is zero, for the coordinate x' is referred to the centre of gravity as origin. Hence, by forming eqn. (11) for the y- and z-coordinates as well, and adding all three, we find

$$(12) \qquad\qquad L = L' + L_0 ,$$

or in words :—The total kinetic energy of a system is composed of the kinetic energy of the motion of the system relative to the centre of gravity, and of the kinetic energy of progression of the mass of the system, assumed to be concentrated at the centre of gravity.

If we define the sum of L' and the internal potential of the system as the *internal energy* of the system, and the sum of L_0 and the external potential as the *external energy*, we have by eqn. (9), calling the former quantity W_i and the latter W_a,

$$(13) \qquad\qquad dW_i + dW_a = 0.$$

Every change in the internal energy must result in an equally large and opposite change in the external energy, and conversely.

It follows that an essential difference exists between the Law of the Conservation of Mechanical Energy and the Principles of the Conservation of Momentum and of Angular Momentum, in so far as the internal forces completely disappear from the final formulæ derived by the methods of integration of these two principles, whereas they are included in eqn. (9) in the expression for V. Further, the third integration method, unlike the other two, does not give three, but only *one* first integral, so that the total number of *first integrals* to be derived by means of the integration principles of mechanics amounts, *at most*, to *seven*. If no external forces are acting on the system, and within it only central forces for which Newton's Third Law of Motion holds, both the vector representing the total momentum and that representing the total angular momentum must remain unchanged in magnitude and direction. Each of these conditions gives rise to three equations, so that, taking into account the energy equation also, we get seven differential equations ; and these no longer contain the second, but only the first derivatives of the coordinates with respect to time, and consequently are of the first order, instead of the second. Of these seven equations, those which express the uniform linear motion of the centre of gravity can be

integrated again; so that, altogether, the integration principles can give *ten integration constants*. Three of these determine the position, and three the velocity of the centre of gravity at a given time; three determine the magnitude and direction of the angular momentum (or, in other words, the absolute value of the latter, and the position of the invariable plane); and one determines the mechanical energy of the system.

The motion of a system of *n* masses can only be completely determined by the simple methods upon which the integration principles are based when *n* is equal to two. For, in that case—the so-called *problem of two bodies*—the space problem can be reduced to one in two dimensions, by choosing as the *x-y*-plane the invariable plane in which, according to the Law of the Angular Momentum, the movement must take place, and then only four first and four second integrals are required. Two of the four first integrals are given by the Law of the Conservation of the Centre of Gravity, one (which expresses the constancy of the rate of describing areas in the above plane) by the Law of the Angular Momentum, and one by the Energy Law. Of the four second integrals two are again obtained from the Law of the Centre of Gravity, while the other two can be derived by *special* methods which are, however, only applicable to the simple problem of two bodies (and into which we shall not enter here).

If the system consists of three masses, *e.g.*, if we consider the motion of the moon under the simultaneous influences of the earth and the sun (as was first done by *Euler*), or the *perturbations* produced in one planet by another during its motion round the sun (which *Laplace* first investigated), a solution based on simple integration methods is no longer possible, for the latter only yield seven in place of the necessary nine first integrals. We have to be content with approximation methods which have been worked out for special cases, whereas the so-called *problem of three bodies*, as a whole, still remains unsolved.

§ 18. The Principle of Virtual Displacements.

In our previous considerations we have dealt with particles, individual or combined in systems, which, although subjected to the influence of certain forces, are of themselves in no way limited as regards their *freedom of motion*.[1]

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[1] *Cf.*, however, § 19, note 2.

The characteristic of a *free* motion of this kind is that for every particle, whether a single individual or a component of a system, the relation

(1) $$\mathbf{K} = m\mathbf{b}$$

is fulfilled, where **K** denotes the resultant of all the forces acting on the particle, and **b** its actual acceleration.

This relation is in general *not satisfied* when the motion of the particle is bound by *definite conditions*, or, as we may say, when it is *not free*. The conditions which limit the freedom of motion may be of two kinds : either a single particle cannot depart from a certain surface [2] or a certain curve, so that it has to move in a completely or partially *prescribed path*, or one particle is *connected* to another.

Since the simple relation given in eqn. (1), by means of which the problem of free motion is solved, is *not satisfied* for restricted motion, physics was confronted with the problem of searching for *more general principles*, to be applied to a system of particles whose freedom of motion may be subjected to arbitrary *limitations*. Such principles must render it possible to determine the actual accelerations produced, and hence the movements of the individual masses, from the given acting forces and the given conditions. Since free motion simply represents a *special* case of restricted motion, it is obvious that these general principles, which we shall now proceed to discuss, must embrace the whole of the earlier treated theory of free motion.

As a starting point for these general principles we shall choose the simplest problem, *viz.*, the question of the *equilibrium* of an *individual particle* whose freedom of motion is limited by certain *conditions*, and which, in spite of the action of a force, remains at rest, or is, as we may say, in equilibrium. If the particle were completely free to move, this would naturally be impossible. The answer given to this question by the principle treated in this section has a certain resemblance to the method of proof which, in logic, is known as *reductio ad absurdum*.

Let us take the case of a particle which, owing to certain conditions limiting its freedom of motion, is in equilibrium in a definite position, despite the action of a force whose magnitude and direction are given. Then, in the absence of such conditions, the force would be able to displace the particle from the given position (in which we may suppose it to be retained up to a certain moment) to a definite

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[2] The surface itself may again be either at rest or in motion.

adjacent position at an infinitesimal distance therefrom,[3] which, however, must naturally be in accord with the given conditions. Now, in doing so, the force must set in motion the particle which was originally at rest ; and since this motion takes place simply and solely under the influence of the force, *kinetic energy* is thereby *produced* and *positive work done*. The particle will hence be in equilibrium when absolutely no infinitesimal displacement from its given position to a neighbouring one (which likewise satisfies the conditions) is possible, in which positive work is done by the force. This is the *necessary* and, moreover, the *sufficient* condition for the existence of equilibrium, and we shall now try to find an *analytical expression* for it.

We define a *virtual displacement*[4] as one in which a particle is displaced from its given actual position to one at an infinitesimal distance therefrom, this position being, however, likewise in accord with the existing conditions. It is a *mental* operation undertaken *by way of trial* in the investigation of the question of equilibrium, whereby it is not intended to say that such a displacement occurs during the motion of the particle considered, or even that it could occur. Inasmuch as this virtual displacement has no connection with the time—in contrast to a displacement which occurs during actual motion, and which represents a portion of the actual path—and since its components are thus not functions of the time, we are *not* able to regard them as *differentials*, as we do the components of the element of the actual path. Consequently, the components of the virtual displacement are to be regarded as only infinitesimally small general alterations, or so-called *variations of the coordinates*, which we may denote by $\delta x, \delta y, \delta z$: the virtual displacement itself may be denoted by δs, seeing that it represents a vector.

Now the work performed in an entirely arbitrary virtual displacement δs by the acting force $\mathbf{K}$ (whose components may be called X, Y, Z) is equal to $\mathbf{K} \, \delta s$, or $(X \, \delta x + Y \, \delta y + Z \, \delta z)$. Hence, we have as the analytical expression of the necessary and sufficient condition for the existence of equilibrium,

$$(2) \qquad X \, \delta x + Y \, \delta y + Z \, \delta z \leqq 0.$$

[3] Only *infinitesimal* displacements may be taken into account, for after a displacement of finite size the necessary conditions for the presence of equilibrium will, in general, no longer be satisfied.

[4] This term is used in contradistinction to the *real* displacements occurring in the case of a movement which either actually happens, or which we may at least assume in our considerations actually to take place.

If this relation holds, it would appear to be deduced *ad absurdum* that the particle can *set itself in motion* under the sole influence of the force **K**, and under the given limitations of its freedom of motion.

If we exclude from our considerations those cases in which it is not possible to have an infinitesimal virtual displacement opposite and equal to any given one,[5] we can only regard every displacement accompanied by the performance of positive work as excluded, when for an entirely *arbitrary*, and hence for *every* arbitrary virtual displacement the *work* is *zero* (seeing that, on reversing the direction of the displacement, the value of the work also changes in sign).[6] The equation

$$(3) \qquad X\, \delta x + Y\, \delta y + Z\, \delta z = 0,$$

therefore, at all events always represents a sufficient condition for equilibrium : it is, however, also necessary when we confine ourselves to those positions in which, to every conceivable virtual displacement, an equal and opposite one is always possible. In what follows this will always be the case.[7]

If we turn our attention from a single particle to a *system of any number of particles*, then we may have included among the conditions limiting the freedom of motion such ones as arise from *connections* between the individual particles. The virtual displacements must then be performed in such a way that the prescribed relations hold between the varied positions just as between the actual positions. Hence the virtual displacements of the individual particles must in general depend on each other ; for only then will the infinitesimally small displacements of the single particles be compatible with the given conditions of the system. Thus there will result a *virtual displacement of the whole system.*

[5] An example of such an exceptional case is afforded by that of a particle which is only free to move in two differently directed planes that intersect each other, the particle being situated exactly in the corner formed by the meeting of the two planes. What is really another exceptional case is that of a heavy body lying on a table ; for, while it can move in any direction vertically or obliquely upwards from its position, displacements equal and opposite to any of these are not possible. They are only possible for horizontal displacements. The motion of a body upon any arbitrary surface, however, can always be treated as though it could not possibly depart from this surface ; and then—apart from quite special exceptional cases—a virtual displacement opposite and equal to any given one is always possible.

[6] For it then becomes

$$X\,(-\,\delta x) + Y\,(-\,\delta y) + Z\,(-\,\delta z) = -\,(X\, \delta x + Y\, \delta y + Z\, \delta z).$$

[7] In eqn. (2) the symbol $<$ may then be omitted.

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The *condition of equilibrium* for the system will thus have to be formulated as follows :—In the case of such a virtual displacement of the whole system, the *sum-total of the work* performed by the acting forces *vanishes*. The condition of equilibrium therefore takes the form [8]

$$(4) \qquad \sum_{h=1}^{h=n} \mathbf{K}_h \, \delta s_h = 0,$$

or,

$$(5) \qquad \sum_{h=1}^{h=n} (X_h \, \delta x_h + Y_h \, \delta y_h + Z_h \, \delta z_h) = 0.$$

The Principle of Virtual Displacements is due to *Johann Bernoulli* (1717). The analytical expression of the principle was the work of *Lagrange*, who recognized in eqn. (5) the *fundamental formula of statics*, and who was readily able to derive from it the most general formula of motion, with the help of the theorem which we shall treat in the next section.[9]

§ 19.   D'Alembert's Principle, and the General Formula of Motion of Lagrange.

If the force acting on a particle which is limited in its freedom of motion does not satisfy the condition [1] laid down by the principle of virtual displacements, the particle cannot be in equilibrium, but must carry out a *restricted motion*. The question as to how this motion takes place, under the given limitations of the freedom of motion, can be reduced, however, to the simpler question already solved in the previous section : *viz.*, under what circumstances the particle can be *in equilibrium* under the given conditions, and subject to the influence of a force acting upon it.

This solution is rendered possible by a principle which is called the *Principle of d'Alembert* after its founder (1743). The fundamental idea underlying this is that the *direction of the actual instantaneous acceleration*, in the case of restricted motion, shall be *distinguished* by the fact that a force, acting

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[8] By $\mathbf{K}_h$ in eqn. (4) is to be understood the resultant of all the external and internal forces which act on the hth particle—a meaning distinct from that in § 15.

[9] The Principle of Virtual Displacements is frequently known as the " Principle of Virtual Velocities." For the reason of this name and for the historical development of the principle, *cf.* the author's "*Grundgleichungen der Mechanik*," lect. 3.

[1] § 18 eqn. (2).

on the particle in this direction, always has the same effect upon the particle as if the latter were *completely free*.[2]

In the case of restricted motion the acting force **K** will, in general, *not* be *equal* to the product $m\mathbf{b}$, either as regards direction or as regards absolute magnitude. We can, however, always *imagine* a force, which may be called the *accessory force*, and which is of such a nature that the vector sum of the acting force **K** and the accessory force (denoted by **P**) is vectorially equal to the product of the mass of the particle and its *actual* acceleration at the moment in question, in the course of its restricted motion. That is to say, we have

$$(1) \qquad\qquad m\mathbf{b} = \mathbf{K} + \mathbf{P}.$$

If, now, instead of the force **K**, a force $(\mathbf{K} + \mathbf{P})$ were to act on the particle under the given conditions, then, in accordance with the fundamental idea mentioned, this force would act upon the particle as if it were entirely *free*, since, according to eqn. (1), the force has the *same direction as the actual acceleration*. The force $(\mathbf{K} + \mathbf{P})$ must therefore impart an acceleration $(\mathbf{K} + \mathbf{P})/m$ to the particle, just as though it were free, and despite the restrictions on its freedom of motion. In accordance with eqn. (1) this acceleration would be equal to the acceleration **b** produced in the particle in its restricted condition by the force **K**. Hence the acceleration of the particle is identical, whether it be due to the force **K** or to the force $(\mathbf{K} + \mathbf{P})$; and, of course, this is only possible when the force **P** is of such a nature that, acting alone on the particle, it would be prevented by the given conditions from producing any acceleration at all.

In a system of particles with restricted freedom of motion, the *accessory forces*, which can be regarded as acting on the individual particles, must be *in equilibrium* at every instant during the motion. This is the real substance of *d'Alembert's Principle*, but the latter is generally expressed

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[2] It must be remembered that this assumption, while extremely *plausible*, is yet only a *postulate*. In simpler cases it is, of course, so obvious that use is made of it without much hesitation. For example, the *restricted* motion carried out by a body under the influence of its weight upon a *plane inclined* at an angle a is regarded as a *free* motion taking place under the influence of a force of magnitude $mg \sin a$, the force acting downwards in the direction of the length of the plane. Of the actual force mg, which acts vertically, only one component, $mg \sin a$, in the direction of the length of the inclined plane, is effective. This follows from the fact that the other component, of magnitude $mg \cos a$, is *non-effective*, since it acts *at right angles* to the plane, and so can produce no alterations in motion along it.

in another form. In order to be able to explain the actual effect of the acting force according to the laws of free motion, we have had to add to this force an imaginary accessory one. This, however, is tantamount to stating that, in consequence of the given limitations of the freedom of motion, the full effect of the acting force **K** is not attained, part of it being *lost*, so that only the residual amount *m*b is effective. This lost force, which may be denoted by **V**, is obviously equal and opposite to the accessory force,[3] for

$$(2) \qquad\qquad \mathbf{V} = \mathbf{K} - m\mathbf{b}.$$

Thus the lost forces are *no more able* to impart an *acceleration* to the system, when they alone are acting on it, than the accessory forces to which, by definition, they are equal and opposite. Hence we can express d'Alembert's Principle in the following form :—The *lost components of the forces* acting on a system must be *in equilibrium at every moment* during the motion.

The accessory and the lost forces exhibit the same mutual behaviour as *actio* and *reactio* in the sense of Newton's Third Law of Motion, and for this reason the lost forces are also known as *reaction forces*. The accessory forces manifest themselves in a *pressure* or a *pull* exerted on the moving particle, with the result that the latter moves otherwise than it would do in the free state, under the sole influence of the acting forces. The lost forces represent the *counter-pressure* or *counter-pull* exerted by the moving masses on the connections or mechanisms that compel them to *deviate* from the paths which, in the free state, they would pursue.[4]

If, now, the lost components of the forces are in equilibrium with each other during the whole movement,

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[3] In speaking of "lost force," we simply mean that **V** must be vectorially subtracted from **K** in order to obtain *m*b : we in no way imply that the absolute amount of the actual acceleration b must always be smaller than K/m.

[4] Since, according to § 4, a particle in a *free* state moves with a constant velocity in a *circular* path, when acted on by a *centripetal force* of magnitude mv^2/r, it is clear that, in the case of a particle following a *prescribed circular path without* any external force acting upon it, the accessory force is likewise mv^2/r, and is directed towards the centre. The *reaction force* (or lost force) is equally large but oppositely directed, and is termed the *centrifugal force*. This relation holds quite generally, if we replace the radius of the circle by the *radius of curvature*. We can, perhaps, best realize a motion in a prescribed path which is not influenced by external forces, by allowing a small ball to roll down an inclined plane into a curved groove cut in a horizontal plate, and as smooth as possible. The movement of a tramcar at uniform speed along a curved track is another example of much the same sort of motion.

then, as *Lagrange* [5] (1788) was the first to recognize, we have only to replace the acting forces in the equation expressing the Principle of Virtual Displacements [§ 18 eqn. (4)] by their lost components in order to obtain a *general formula of motion*. This formula is consequently given by

$$(3) \qquad \sum_{h=1}^{h=n} \left\{ (\mathbf{K}_h - m_h \, \mathbf{b}_h) \, \delta \mathbf{s}_h \right\} = 0,$$

or, analytically, we may also write

$$(4) \qquad \sum_{h=1}^{h=n} \left\{ \left(X_h - m_h \frac{d^2 x_h}{dt^2} \right) \delta x_h + \left(Y_h - m_h \frac{d^2 y_h}{dt^2} \right) \delta y_h \right.$$
$$\left. + \left(Z_h - m_h \frac{d^2 z_h}{dt^2} \right) \delta z_h \right\} = 0.$$

§ 20. Deduction of the Special Equations of Motion from the General Formula of Motion of Lagrange.

The *dynamic general formula of Lagrange* which was derived in the above section [§ 19 eqn. (4)] does not merely express a general property of any arbitrary movement, for it has the great advantage that, by certain methods likewise devised by *Lagrange*, it is possible to deduce from it the so-called *special equations of motion of the system* for any particular case, *i.e.*, the equations which describe the motions of the individual particles forming the system. These equations appear, as it were, *compounded* in the general formula, while the latter, on suitable treatment, can be again *resolved* into them.

We shall assume that the freedom of motion of the system is limited by *m* equations between the 3*n* coordinates of the system.[1] Naturally, the forces do not occur in these

[5] D'Alembert himself utilized the *special* conditions of equilibrium of the system under consideration for the solution of every dynamical problem : *e.g.*, in the case of a lever, the lost components of the forces must always be inversely proportional to the corresponding arms of the lever.

[1] *Heinrich Hertz* was the first to draw attention to the importance of the fact that the conditions restricting a system's mobility can be very well expressed in equations between the *differentials* of the coordinates, without it being necessary for these equations to be integrable. In that case the conditions naturally cannot be expressed by means of equations between the coordinates themselves. In accordance with *Hertz* such conditions are termed *non-holonomous* : in like manner a system is called non-holonomous even when it is subject to only one non-holonomous condition. The most usual example of such a system is that of a *sphere rolling* on a plane surface without slip. *Hölder* was the first subsequently to show that non-holonomous systems obey the laws of classical mechanics, a fact which *Hertz* at first doubted.   In the following considerations it will always be taken for granted that the system in question is holonomous, as was the general practice until *Hertz's* time.

equations, and in what follows we shall only consider the comparatively simple case that these *equations of condition do not contain the time*.[2] The m equations can then be reduced to the form [3]

(1) $G_i(x_1, y_1, z_1, x_2 \ldots \ldots y_n, z_n) = 0, \quad (i = 1 \text{ to } m).$

The number $(s = 3n - m)$ we define as the *number of degrees of freedom* of the system under consideration.   Thus a completely free system has just as many degrees of freedom as coordinates, whereas the number of degrees of freedom is diminished by one for every equation of condition introduced. If, therefore, there are as many independent equations of condition as coordinates, no motion of the system is possible, for the values of all the coordinates are then given in advance, and so cannot be altered.

In a system of s degrees of freedom we can now select any s of the $3n$ coordinates, and denote these by ξ_k $(k = 1 \text{ to } s)$.[4]   We can regard these s coordinates as completely *independent* of each other, while, by means of the m equations of condition, it must be possible to express the remaining m coordinates in terms of the s independent coordinates.   We can, however, also express the variations of the m dependent coordinates by those of the s independent ones.   For, if $\varphi(u, v, w)$ be any function whatsoever of any given variables u, v, w, it is clear that the variation $\delta\varphi$ in the function φ, due to the variations $\delta u, \delta v, \delta w$ of its arguments, is equal to the sum of those variations which would be produced in φ by the variation of u by itself, or of v by itself, or of w by itself.   Now the ratio between a variation in φ due to a variation in u alone, v and w remaining constant, and the corresponding variation in u, is simply the partial derivative of φ with respect to u.   Hence if u varies by δu, v and w remaining constant, φ will vary by the amount $(\partial\varphi/\partial u).\delta u$, owing to the functional connection; so that, in general, a variation in φ is given by

(2) $\delta\varphi(u, v, w) = \dfrac{\partial\varphi}{\partial u}\,\delta u + \dfrac{\partial\varphi}{\partial v}\,\delta v + \dfrac{\partial\varphi}{\partial w}\,\delta w.$

Applying this formula to the equations (1), in which the $G_i\,(i = 1 \text{ to } m)$ represent functions of the coordinates, we

[2] In that case the system is called *scleronomous*, while we speak of a *rheonomous* system if the equations of condition contain the *time* explicitly.

[3] It is, of course, quite unnecessary that each of these m equations should contain all $3n$ coordinates.

[4] We shall therefore use the symbol ξ in general also for the y- and z-coordinates.

obtain by this so-called variation of eqn. (1) m new relations of the form

$$(3) \quad \sum_{h=1}^{h=n} \left\{ \frac{\partial G_i}{\partial x_h} \delta x_h + \frac{\partial G_i}{\partial y_h} \delta y_h + \frac{\partial G_i}{\partial z_h} \delta z_h \right\} = 0, \quad (i = 1 \text{ to } m).$$

By means of these equations (3), let us now replace the variations of the m dependent coordinates in the general formula of motion [§ 19 eqn. (4)] by the variations of the s independent ones. Then, arranging the separate members in the proper order, the general formula of motion reduces to the form

$$(4) \quad A_1 \delta \xi_1 + A_2 \delta \xi_2 + \ldots + A_s \delta \xi_s = 0 ;$$

where the quantities A_1, A_2, etc., are functions of the forces and of the coordinates, as well as of the time-derivatives of the coordinates, but not of the virtual displacements : in other words, the sum to be put equal to zero must always be a *linear* function of the variations of the independent coordinates. Now, since the s coordinates ξ_k are independent of each other, we are obviously completely at liberty to vary any one independent coordinate, so that it increases or decreases at will, without being in any way concerned with the variation of the remaining $(s - 1)$ values ξ_k. We can, therefore, entirely at will make each of the s variations positive or negative, independently of the others. If in doing this we always give $\delta \xi_k$ the same sign as the quantity A_k, we can so arrange that no single summand on the left-hand side of eqn. (4) is negative. Since, however, the sum of terms, none of which is negative, can only be zero if *each* individual summand itself vanishes, the following equations must *always* be satisfied (for $\delta \xi_k$ is assumed always to differ from zero) :—

$$(5) \quad A_1 = 0, A_2 = 0, \ldots \ldots A_s = 0.$$

The formation of these equations gives rise to s equations of motion, that is, to just as many as there are degrees of freedom in the system, and therefore sufficient for the complete description of the motion of the latter. The equations of motion for the remaining m coordinates can then be obtained from the equations which express the connection between these m coordinates and the s independent coordinates.[5]

[5] We see, also, that for a completely free system the general formula of motion reduces, with this method of treatment, to the equations of motion of a free particle :

$$X_h = m_h \frac{d^2 x_h}{dt^2}, \text{ etc.}$$

Thus the general formula of motion retains its validity also in this case, though, naturally, it has then no particular value.

Frequently a more practical process for the derivation of the special equations of motion than the one just considered is that of the *method of undetermined multipliers*, which we likewise owe to *Lagrange*. In this method, we suppose each of the m equations (3) to be multiplied by a factor (λ_i) to which no definite value is at first ascribed, and the resulting equations to be added to the general formula of motion. In this way we obtain an equation of the form

$$
(6) \quad
\left\{
\begin{aligned}
&\sum_{h=1}^{h=n} \left\{ \left(X_h - m_h \frac{d^2 x_h}{dt^2} + \sum_{i=1}^{i=m} \lambda_i \frac{\partial G_i}{\partial x_h} \right) \delta x_h \right. \\
&\quad + \left(Y_h - m_h \frac{d^2 y_h}{dt^2} + \sum_{i=1}^{i=m} \lambda_i \frac{\partial G_i}{\partial y_h} \right) \delta y_h \\
&\quad + \left. \left(Z_h - m_h \frac{d^2 z_h}{dt^2} + \sum_{i=1}^{i=m} \lambda_i \frac{\partial G_i}{\partial z_h} \right) \delta z_h \right\} = 0.
\end{aligned}
\right.
$$

The equation thus contains altogether $3n$ expressions in brackets, which are multiplied by the variations of the $3n$ coordinates of the system. The values possessed by these bracketed expressions depend on those of the m multipliers, which latter have so far been left undetermined. We can hence make the *postulate* that these m multipliers shall be of such a nature that m out of the $3n$ bracketed expressions contained in eqn. (6) become *zero*.

Of the $3n$ terms forming the left-hand side of eqn. (6) there are then only $(3n - m)$ left, in which definite expressions in brackets are multiplied by the variations of $(3n - m)$ coordinates. Since on the other hand, however, the system possesses $(3n - m)$ degrees of freedom, the variations of the remaining $(3n - m)$ coordinates can be regarded as completely independent of each other; whence it follows that each of the remaining $(3n - m)$ expressions in brackets must *vanish individually*.

Moreover, m expressions in brackets in the original eqn. (6) are equal to zero, in consequence of the postulate laid down regarding the nature of the multipliers; and so we obtain the special equations of motion for the n particles of the system by simply equating to zero *each* of *all* the $3n$ expressions in brackets in eqn. (6). It is true that the resulting $3n$ equations contain $(3n + m)$ unknowns, *viz.*, the coordinates (as functions of the time) and, in addition, the multipliers; but, as we have at our disposal yet other m equations, in the form of the equations of condition, (1), we can also determine the multipliers from these $(3n + m)$

equations. Having found these values, we have only to insert them in the bracketed expressions in eqn. (6), equating these latter to zero, in order to obtain for any given particle the special equations of motion in the form

(7)
$$
\begin{cases}
m_h \dfrac{d^2 x_h}{dt^2} = X_h + \displaystyle\sum_{i=1}^{i=m} \lambda_i \dfrac{\partial G_i}{\partial x_h}, \\[2ex]
m_h \dfrac{d^2 y_h}{dt^2} = Y_h + \displaystyle\sum_{i=1}^{i=m} \lambda_i \dfrac{\partial G_i}{\partial y_h}, \\[2ex]
m_h \dfrac{d^2 z_h}{dt^2} = Z_h + \displaystyle\sum_{i=1}^{i=m} \lambda_i \dfrac{\partial G_i}{\partial z_h}.
\end{cases}
$$

These relations are known as *Lagrange's Equations of Motion of the First Form* to distinguish them from the formulæ to be discussed in § 22. Comparing them with § 19 eqn. (1), we see at once that the sum

$$
\sum \lambda_i \frac{\partial G_i}{\partial x_h}
$$

together with the two analogous summations in the other two equations represent the components of the *accessory force*, which we have to imagine as acting on the hth particle, in addition to the force X_h, Y_h, Z_h, in order to be able to explain restricted motion in accordance with the principles of free motion.

§ 21. Hamilton's Principle.

The fundamental formula of equilibrium can be obtained by comparing the actual position of a system with a *neighbouring position deviating* by an infinitesimal amount from it, and likewise compatible with the given conditions. Thus the idea suggests itself of comparing the actual *motion* of a system with one which deviates infinitesimally from it and likewise satisfies the conditions prescribed for the system, and which, just like *virtual* displacements, is purely an imaginary process. With every point along the actual path there will, in general, be *associated* an adjacent point at an infinitesimal distance therefrom, the latter point resulting from a vanishingly small *variation* of the former's *coordinates*, such variation being consistent with the given conditions : by joining up these associated points we find the *varied path*. It is, of course, also possible to make any arbitrary assumptions as to the manner in which this varied path is described, but we arrive at particularly simple relations by introducing the special hypothesis that the

time shall remain *unaffected by the variation*. That is to say, whatever may be the time-interval taken in the actual motion for the passage between any two points, the time required to traverse the interval between the corresponding two associated points in the case of the varied motion shall be equally great.

Let A be a point on the actual path, its coordinates being x, y, z; let B be the point which the moving particle reaches, during the course of the actual motion, after the expiration of an interval dt from the moment of its passing through A; and let A' be the point on the varied path associated with A. Then the x-coordinate of the point B, which we may call x_2, is given by the relation

$$(1) \qquad x_2 = x + \frac{dx}{dt}\,dt.^1$$

On the other hand, the x-coordinate of the point A', which we shall call x_3, is given by

$$(2) \qquad x_3 = x + \delta x.$$

At the end of the interval dt, the particle reaches the point B', during the course of the varied motion; if x_4 be the x-coordinate of this point, we have

$$(3) \qquad x_4 = x_3 + \frac{dx_3}{dt}\,dt = x + \delta x + \frac{dx}{dt}\,dt + \frac{d}{dt}\,(\delta x)\,dt.$$

Since, however, we have stipulated that the time shall be unaffected by the variation, B' must be identical with the point associated with B; and hence

$$x_4 = x_2 + \delta x_2,$$

or, by eqn. (1),

$$(4) \qquad x_4 = x + \frac{dx}{dt}\,dt + \delta x + \delta\left(\frac{dx}{dt}\,dt\right).$$

Now, if dt is independent of the variation,

$$\delta\left(\frac{dx}{dt}\,dt\right) = \delta\left(\frac{dx}{dt}\right)dt\,;$$

whence, comparing eqns. (3) and (4), we find

$$(5) \qquad \frac{d}{dt}\,(\delta x) = \delta\left(\frac{dx}{dt}\right).$$

This simple relation, together with the analogous relations for y and z, only holds when the time is unaffected by the variation. Where eqn. (5) is satisfied, however, we can

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[1] For, if v_x is the x-component of the velocity, $x_2 = x + v_x\,dt$.

easily deduce from Lagrange's general formula of motion the *variation of the kinetic energy, i.e.*, the difference that exists between the values of the kinetic energy according as the particle passes through a point on the actual path, or through the associated point on the varied path.

Lagrange himself indicated the simplest method of obtaining an expression for the variation of the kinetic energy, *viz.*, by making a *complete differential* out of the product

$$\frac{d^2x}{dt^2}\,\delta x$$

occurring in the general formula of motion. We have

(6) $$\frac{d^2x}{dt^2}\,\delta x = \frac{d}{dt}\left(\frac{dx}{dt}\,\delta x\right) - \frac{dx}{dt}\frac{d}{dt}\,(\delta x).$$

By eqn. (5), however,[2]

(7) $$\frac{dx}{dt}\frac{d}{dt}(\delta x) = \frac{dx}{dt}\,\delta\left(\frac{dx}{dt}\right) = \delta\left[\frac{1}{2}\left(\frac{dx}{dt}\right)^2\right].$$

Let us now form eqn. (6) for all the coordinates of the system, multiply each of these equations by the corresponding mass, and then add all $3n$ equations together. In this way we obtain on the left-hand side the sum

$$\sum_{h=1}^{h=n}(X_h\,\delta x_h + Y_h\,\delta y_h + Z_h\,\delta z_h).$$

It is easy to see that, when a potential is present, this sum simply represents the *variation of the potential* taken with the opposite sign. For, by § 20 eqn. (2),

$$\delta V = \sum_{h=1}^{h=n}\left(\frac{\partial V}{\partial x_h}\delta x_h + \frac{\partial V}{\partial y_h}\delta y_h + \frac{\partial V}{\partial z_h}\delta z_h\right),$$

and

$$X_h = -\frac{\partial V}{\partial x_h},\ \text{etc.}$$

On the other hand, if we multiply eqn. (6) by the mass and take the sum over all the coordinates, the third (negative) term represents [according to eqn. (7)] the variation of the kinetic energy (L), taken with a negative sign. Hence, on

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[2] For, by § 20 eqn. (2),

$$\delta(u^2) = \frac{\partial(u^2)}{\partial u}\,\delta u = 2u\,\delta u,$$

and so

$$u\,\delta u = \delta\left(\frac{1}{2}u^2\right).$$

the left-hand side of the summational equation we have $(-\delta V)$, and on the right-hand side $(-\delta L)$. Thus, the equation obtained by multiplying eqn. (6) by the mass, and by taking the sum over all $3n$ coordinates, can be written in the form

$$(8) \quad \delta (L - V) = \frac{d}{dt}\left\{\sum_{h=1}^{h=n} m_h \left[\frac{dx_h}{dt}\delta x_h + \frac{dy_h}{dt}\delta y_h + \frac{dz_h}{dt}\delta z_h\right]\right\}.$$

This equation assumes an especially simple form on the supposition that the variation is not only independent of the time, but that it is also of such a nature that, for two given times (which we may call t_0 and t_1), the *position* of the system for the actual motion *coincides* with that for the varied. We may term t_0 and t_1 the *initial* and the *final times*, and the corresponding positions the *initial* and the *final positions*, without in any way meaning to imply that the motion only begins at the time t_0 and ends at the time t_1. We simply mean that the *portion* of the movement selected for the imaginary variation is *bounded* by the times t_0 and t_1. Bearing these *hypotheses* in mind, let us multiply eqn. (8) by dt, and then *integrate* between the times t_0 and t_1, thereby obtaining on the right-hand side the difference in the values assumed by the expression in curly brackets, in eqn. (8), for the times t_0 and t_1. In accordance, however, with our hypothesis, the variations of all the coordinates at the initial and final times must be equal to zero, so that the expression in curly brackets is itself zero both at the time t_0 and at the time t_1. Hence we find the simple relation

$$(9) \qquad \int_{t_0}^{t_1} \delta (L - V)\, dt = 0.$$

Therefore the time-integral of the variation of the difference between kinetic and potential energy vanishes between two time-limits for which the position of the system is not varied, provided that the variation of the motion be *independent of the time*. This theorem is known as *Hamilton's Principle*, after its discoverer (1834).[3]

[3] Since the variation is taken as independent of the time, we can interchange the symbols of variation and integration, in accordance with the principles of the calculus of variations; so that we may write eqn. (9) in the form

$$(10) \qquad \delta \int_{t_0}^{t_1}(L - V)\, dt = 0.$$

Now any definite integral whose variation vanishes represents a *stationary value*, by the rules of the calculus (into which we shall not further enter here); *i.e.*, it is either a maximum or minimum, or a so-called horizontal point of inflection. Hence, using *Helmholtz's* definition of the difference $(L - V)$ as the *kinetic potential*, eqn. (10), divided by the difference $(t_1 - t_0)$, can be

§ 22. The Generalized and Canonical Equations of Motion.

The equation which expresses Hamilton's Principle resembles the dynamic general formula of Lagrange in that it represents only a *symbolical formula* : it does not lead to new integrals of the equations of motion as do the Principles of the Conservation of the Centre of Gravity, of the Conservation of the Total Angular Momentum, and of the Conservation of Energy. Its merit, however, consists in the fact that, when treated in a certain way, it is immediately *resolved* into the special equations of motion of the system, just as is Lagrange's general formula of motion. Moreover, in this respect, Hamilton's Principle has a great advantage over Lagrange's formula, for it contains nothing whatever that is in any way connected with a definite coordinate system, especially as it contains no vector quantities, but *only scalars*.

It is obviously possible to deduce Lagrange's general

expressed in the following form (also due to Helmholtz) :—Among all *conceivable* motions which transport a system in a given time from a given initial into a given final position, that *motion* will *actually* occur, for which the *average value* of the *kinetic potential* represents a *stationary value.*

Where a potential exists, the general formula of equilibrium [§ 18 eqn. (4)] can also be replaced, in consequence of § 20 eqn. (2), by

(11) $$\delta V = 0 \; ;$$

in other words, a system of particles will be *in equilibrium* when its *potential energy* has a *limiting value.* We speak of *unstable* or of *stable equilibrium* according as this limiting value represents a *maximum* or a *minimum*. If δV vanishes also for a finite displacement, the equilibrium is termed *indifferent.* The best known example of an indifferent equilibrium is that of a heavy sphere resting on a horizontal surface.

The attempts to *deduce* the special laws of motion for a system from *maximum* or *minimum* properties, which should *distinguish* the *actual motion* from all other *conceivable* motions compatible with the given conditions, reach back to the far ages. Thus, Heron, of Alexandria, quite correctly deduced the equality of the angles of incidence and reflection from the postulate that the path which a ray of light traverses between two points, when it is reflected from a plane surface, is the one for which the *shortest* time is necessary. In the eighteenth century *Maupertuis* endeavoured to formulate a supreme law of nature in such a form that, in all natural processes, the " *action* " (measured by the product of the path, time, and velocity) must be a *minimum*. *Euler* and *Lagrange* gave an exact basis and form to this principle, its conception and application by Maupertuis being extremely superficial. By means of this so-called *Principle of Least Action* (which we cannot further discuss here), Hamilton, in 1834, established his principle, for which (as contrasted with that of Euler and Lagrange) the independence of the variation of the time is an essential condition. *Cf.* the author's "*Grundgleichungen der Mechanik*," lect. 19 and 20.

It may just be mentioned that Hamilton's Principle also holds good in the absence of a potential, provided that $\delta'V$ be symbolically written for δV, where by $\delta'V$ is understood the sum

$$\sum (X_h \, \delta x_h + Y_h \, \delta y_h + Z_h \, \delta z_h),$$

taken with the opposite sign.

formula of motion from Hamilton's Principle by simply
reversing the method used in the previous section, but such
a roundabout way is quite unnecessary in order to obtain
the special equations of motion from Hamilton's Principle.

In order to demonstrate this we shall make use of so-called
generalized coordinates in the derivation, in what follows, of
the special equations of motion.   By these we mean arbitrary
variables which represent, not necessarily merely lengths,
but also angles, surfaces, volumes, *etc.*, though they must
be of such a nature that the $3n$ orthogonal coordinates of
the system can be completely expressed by means of them.[1]
If, in particular, we make their number equal to that of the
degrees of freedom of the system, then these s generalized
coordinates can be regarded as *completely independent of
each other*.   Hence, when we are able to establish a
differential equation for each of the generalized coordinates,
the motion of the system is completely determined by these
s so-called *generalized equations of motion*.

We must naturally be able to deduce the generalized
equations of motion from Lagrange's general formula of
motion by expressing the rectangular coordinates, together
with their variations and time-derivatives, in terms of the
generalized coordinates, together with their variations and
derivatives.   As a matter of fact, that is the way in which
these equations were obtained by Lagrange, who created
the idea of generalized coordinates.   In the following
paragraphs, however, we shall deduce them from Hamilton's
Principle, because, on the one hand, this way is much
simpler, and, on the other, because the method of obtaining the
equations of the actual motion from the symbolical formula
of Hamilton's Principle is at the same time demonstrated.

Let q_i ($i = 1$ to s) denote the s generalized coordinates
of the system ; then the orthogonal coordinates of any one
of the n particles can be represented as definite functions of
the generalized coordinates, so that we have, in general,
relations of the form

(1) $$x_h = \varphi_h (q_1, q_2, q_3 \ldots q_s).$$

Now, in order to express the kinetic energy in Hamilton's
equation in terms of the generalized coordinates, we must
differentiate these $3n$ equations (1) with respect to time,
then square them and multiply them by the appropriate
masses, and finally add them all together.   Denoting the

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[1] The generalized coordinates are in no way bound to be all of the same
dimensions ; some may be lengths, others angles, *etc.*

time-derivative of a quantity by a dot placed over the letter representing it (in accordance with custom), we find, by differentiating eqn. (1) with respect to time,

$$(2) \qquad \dot{x}_h = \frac{\partial x_h}{\partial q_1}\dot{q}_1 + \frac{\partial x_h}{\partial q_2}\dot{q}_2 + \ldots + \frac{\partial x_h}{\partial q_s}\dot{q}_s.$$

Squaring this equation we have

$$(3) \quad \begin{cases} \dot{x}_h^2 = \left(\frac{\partial x_h}{\partial q_1}\right)^2 \dot{q}_1^2 + \left(\frac{\partial x_h}{\partial q_2}\right)^2 \dot{q}_2^2 + \ldots + \left(\frac{\partial x_h}{\partial q_s}\right)^2 \dot{q}_s^2 \\[2mm] \quad + 2\frac{\partial x_h}{\partial q_1}\frac{\partial x_h}{\partial q_2}\dot{q}_1\dot{q}_2 + 2\frac{\partial x_h}{\partial q_1}\frac{\partial x_h}{\partial q_3}\dot{q}_1\dot{q}_3 + \ldots \\[2mm] \quad + 2\frac{\partial x_h}{\partial q_2}\frac{\partial x_h}{\partial q_3}\dot{q}_2\dot{q}_3 + \ldots \end{cases}$$

It is easy to see that we can write this in the form

$$(4) \qquad \dot{x}_h^2 = \sum_{i=1}^{i=s}\sum_{k=1}^{k=s}\frac{\partial x_h}{\partial q_i}\frac{\partial x_h}{\partial q_k}\dot{q}_i\dot{q}_k;$$

for, putting k equal to i, we find the terms in the first row of eqn. (3); whereas, if i and k are different, every term occurs twice.

Forming eqn. (4) for each of the $3n$ orthogonal coordinates, multiplying by the corresponding masses, and then adding all these equations together, we have, since the sum on the left-hand side gives twice the kinetic energy

$$(5) \qquad 2L = \sum_{i=1}^{i=s}\sum_{k=1}^{k=s} c_{ik}\dot{q}_i\dot{q}_k,$$

where

$$c_{ik} = \sum_{h=1}^{h=n} m_h \left[\frac{\partial x_h}{\partial q_i}\frac{\partial x_h}{\partial q_k} + \frac{\partial y_h}{\partial q_i}\frac{\partial y_h}{\partial q_k} + \frac{\partial z_h}{\partial q_i}\frac{\partial z_h}{\partial q_k}\right].$$

Thus the coefficients c_{ik} depend only on the values of the generalized coordinates [by eqn. (1)], but are completely independent of the values of their time-derivatives: the latter may be suitably termed *generalized velocities* (and denoted by the symbol $\dot{q}$).

In accordance with eqn. (5) the kinetic energy represents a homogeneous quadratic function of the generalized velocities, and it can therefore be regarded as a function of $2s$ variables $q_1, q_2 \ldots q_s, \dot{q}_1, \dot{q}_2 \ldots \dot{q}_s$. Consequently, the variation of the kinetic energy which has to be inserted in Hamilton's equation is given by

$$(6) \qquad \delta L = \sum_{i=1}^{i=s}\left(\frac{\partial L}{\partial q_i}\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\delta\dot{q}_i\right).$$

The potential, on the other hand, since it depends only on the position of the system, but not on the time, will be completely determined by the s generalized coordinates themselves (for the system has only s degrees of freedom); it will, however, be entirely independent of the generalized velocities. Hence

$$(7) \qquad \delta V = \sum_{i=1}^{i=s} \frac{\partial V}{\partial q_i} \, \delta q_i.$$

Let us now insert the values given by eqns. (6) and (7) in Hamilton's equation, at the same time making use of § 21 eqn. (5), which must be satisfied if Hamilton's Principle is to be applied. Since we can place the summation symbol in front of the integration symbol, we have

$$(8) \qquad \sum_{i=1}^{i=s} \int_{t_0}^{t_1} \left\{ \left(\frac{\partial L}{\partial q_i} - \frac{\partial V}{\partial q_i} \right) \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \frac{d}{dt} (\delta q_i) \right\} dt = 0.$$

We can *integrate by parts* the last term on the left-hand side by forming from it a complete differential; for

$$(9) \qquad \frac{\partial L}{\partial \dot{q}_i} \frac{d}{dt} (\delta q_i) = \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \delta q_i \right) - \delta q_i \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right);$$

whence, multiplying by dt, and integrating between t_0 and t_1,

$$(10) \qquad \int_{t_0}^{t_1} \frac{\partial L}{\partial \dot{q}_i} \frac{d(\delta q_i)}{dt} \, dt = \left| \frac{\partial L}{\partial \dot{q}_i} \delta q_i \right|_{t_0}^{t_1} - \int_{t_0}^{t_1} \delta q_i \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) dt.$$

If, now, the conditions necessary for the applicability of Hamilton's equation are satisfied, the first term on the right-hand side must vanish, since, by hypothesis, the coordinates are to remain unvaried for t_0 and t_1. Eqn. (8) therefore takes the form

$$(11) \qquad \sum_{i=1}^{i=s} \int_{t_0}^{t_1} \left[\frac{\partial L}{\partial q_i} - \frac{\partial V}{\partial q_i} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) \right] \delta q_i \, dt = 0.$$

Since the generalized coordinates were assumed to be completely independent of one another, we can choose any one variation δq_i entirely at will and independently of the variations of the other generalized coordinates. Hence, by suitable choice of the δq_i, we can always prevent the whole expression following the integral sign (the terms in square brackets multiplied by $\delta q_i \, dt$) from becoming negative for the ith generalized coordinate, even for one of the time intervals over which the integration is performed. As δq_i need never be zero, eqn. (11) can consequently only be satisfied if for *every* i the expression in square brackets is *individually* equal to *zero*.

In this way we find the *s special equations of motion* of the system in the form :

$$(12) \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = -\frac{\partial V}{\partial q_i} \ (i = 1 \text{ to } s),$$

and these are the so-called *Generalized Equations of Motion of Lagrange* or, by way of distinction from § 20 eqn. (7), Lagrange's Equations of Motion of the *second form.*

Since the rectangular components of the forces acting on the system are represented by the partial derivatives of the potential with respect to the rectangular coordinates, with the sign reversed, the partial derivatives of the potential with respect to the generalized coordinates, taken with reversed sign, are also termed the *generalized components of force* or the generalized forces.[2] On the other hand, the partial derivatives of the kinetic energy with respect to the generalized velocities are termed the *generalized momenta* of the system.[3]

If we differentiate eqn. (5) with respect to a definite generalized velocity, *e.g.*, $\dot{q}_h$, the partial derivative of every term in the double summand for which *i and k* differ from *h* will vanish, whereas, for those terms in which *i or k* is equal to *h*, the same value will occur *twice.* Thus we obtain the relation

$$(13) \qquad 2\frac{\partial L}{\partial \dot{q}_i} = 2\sum_{k=1}^{k=s} c_{ik}\,\dot{q}_k,$$

and it follows from eqn. (5) that

$$(14) \qquad 2L = \sum_{i=1}^{i=s} \frac{\partial L}{\partial \dot{q}_i}\,\dot{q}_i,$$

or, denoting the *generalized momenta* by p_i, that

$$(15) \qquad 2L = \sum_{i=1}^{i=s} p_i\,\dot{q}_i.$$

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[2] It must, however, be remembered that generalized forces by no means need to have the dimensions of a mechanical force. It is solely necessary that the product of a generalized force and the corresponding generalized coordinate should have the dimensions of *work.* If, for example, the generalized coordinate is an angle (*i.e.*, a quantity of no dimensions), the corresponding generalized force must possess the dimensions of a couple ; if the independent coordinate is a volume magnitude, the corresponding generalized force will have the dimensions of a pressure, *etc.*

[3] This term is clear from the particular relation

$$m\dot{x} = \frac{\partial L}{\partial \dot{x}}.$$

As this equation shows, the kinetic energy is thus completely determined by the generalized coordinates and momenta. On the other hand, however, as we saw from the earlier eqn. (6), it may also be regarded as a complete function of the generalized coordinates and velocities. We may write both

$$(16) \qquad L = F(q_i, \dot{q}_i)$$

and

$$(17) \qquad L = G(p_i, q_i).$$

Since p_i is equal to $\partial F / \partial \dot{q}_i$, we have thus on the one hand

$$(18) \qquad dL = \sum \left\{ \frac{\partial F}{\partial q_i} dq_i + p_i \, d\dot{q}_i \right\},$$

and on the other hand,

$$(19) \qquad dL = \sum \left\{ \frac{\partial G}{\partial p_i} dp_i + \frac{\partial G}{\partial q_i} dq_i \right\}.$$

Adding these two equations together, we have

$$(20) \qquad 2 \, dL = \sum \left\{ \left(\frac{\partial F}{\partial q_i} + \frac{\partial G}{\partial q_i} \right) dq_i + p_i \, d\dot{q}_i + \frac{\partial G}{\partial p_i} dp_i \right\}.$$

Again, it follows from eqn. (15) that

$$(21) \qquad 2 \, dL = \sum \left\{ p_i \, d\dot{q}_i + \dot{q}_i \, dp_i \right\}.$$

Subtracting eqn. (21) from eqn. (20), we obtain thus

$$(22) \qquad 0 = \sum \left\{ \left(\frac{\partial F}{\partial q_i} + \frac{\partial G}{\partial q_i} \right) dq_i + \left(\frac{\partial G}{\partial p_i} - \dot{q}_i \right) dp_i \right\}.$$

Now, as we have seen, the variables, q_i and p_i are to be regarded as mutually independent, so that eqn. (22) can be satisfied only when each of the two expressions in brackets separately vanishes. Hence we have the two equations

$$(23) \qquad \frac{\partial G}{\partial q_i} = - \frac{\partial F}{\partial q_i}$$

and

$$(24) \qquad \dot{q}_i = \frac{\partial G}{\partial p_i}.$$

Since, in eqn. (12), the kinetic energy L was regarded as a function of q and $\dot{q}$, thus, as the function F, we can, by introducing the function G and using eqn. (23), write eqn. (12) in the form

$$(25) \qquad \frac{dp_i}{dt} = - \frac{\partial(G + V)}{\partial q_i} \quad (i = 1 \text{ to } s).$$

The sum $(G + V)$, however, simply represents the *energy*, provided that this be regarded as a function of q and p. This function of the generalized coordinates and generalized momenta which represents the energy is known as the *Hamiltonian function*. Denoting it by the symbol H, we have, inasmuch as the potential depends *only* on the co-ordinates,

$$(26) \qquad H = G(q,p) + V(q),$$

and therefore

$$(27) \qquad \frac{\partial H}{\partial p_i} = \frac{\partial G}{\partial p_i}.$$

From eqns. (24) and (25) follow the so-called *canonical equations of motion* in the form

$$(28) \qquad \frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}.$$

Chapter III

THE MOTION OF RIGID BODIES

§ 23.   The Conception of a Rigid Body.

THE *conception of a material point*, which formed the basis of our previous considerations, provides the *simplest* form in which we can discuss the motion of bodies which, as they occur in nature, are *really solid.*   It follows that this only suffices for simple cases from the elementary fact of experience that, in a solid body, it is quite possible for some individual parts to be at rest while others are in motion. In our considerations of the motion of solid bodies we may most readily take into account the necessity of imagining a solid body to be *composed of separate parts*, by substituting the conception of a *system of discrete particles* for that of the simple particle.   At the same time we attribute a *special property* to this system, which corresponds to the most important property shown, in greater or less approximation, by actual solid bodies.   This arises from the fact that the *form* and internal distribution of mass in solid bodies *vary*, in general, *so little* during motion, and under the action of forces, that such variations can be *neglected* for the purposes of many dynamical investigations.   We accordingly *define* a *rigid body* as a *system of particles* whose *mutual distances* are *invariable*.[1]   Denoting the quite arbitrary number of the particles composing the body by n, it follows that the freedom of motion of the system is limited by equations of condition of the form

$$(1) \qquad (x_k - x_h)^2 + (y_k - y_h)^2 + (z_k - z_h)^2 = a_{hk}{}^2 \, ;$$

where a_{hk} signifies the *constant* distance between the hth and the kth particle whose coordinates, with respect to any given coordinate system, are x_h, y_h, z_h and x_k, y_k, z_k respectively.

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[1] The fact that we can in this way obtain a *picture* of the mechanical behaviour of solid bodies, which is sufficient for many purposes, has nothing whatever to do with the question of the molecular structure of matter.   It is, indeed, quite unnecessary that the picture should correspond to the reality : it is sufficient that the results deduced therefrom more or less exactly agree with actual observations.

Now although $n(n-1)/2$ equations of the form of eqn. (1) can be established, it in no way follows that all these equations are *independent* of each other. We can easily see this by selecting *three* of the n particles [2] and constructing a *coordinate system* through them in such a way that the first particle forms the origin, the line joining the first and second, the x-axis, and the plane determined by the three points, the x-y-plane. It is obvious that the mutual distances of all the particles are determined as soon as we know the coordinates of all the n particles with respect to this system. For this, however, we do not require $3n$ numerical data, but only $(3n-6)$, since, in consequence of the way in which the coordinate system was constructed, *six coordinates always vanish*, viz., the three coordinates of the first particle, the y- and z-coordinates of the second, and the z-coordinate of the third particle. Hence, even if we express the equations in terms of a coordinate system outside of the body, it is nevertheless not possible to establish more than $(3n-6)$ mutually *independent* equations of the form of eqn. (1); the additional ones that we may write down can be derived from these other $(3n-6)$. A rigid body whose freedom of motion is only limited by the rigid connections between its parts thus possesses *six degrees of freedom*.

The number of degrees of freedom may, of course, be still further limited. For example, if one point of the rigid body is *fixed* so that its coordinates are invariable—we then speak of a *rotation* of the body about this point—the $(3n-6)$ equations of condition, arising from the rigidity of the body, are increased in number by three others quite independent of them : these express the constancy of the three coordinates of the fixed point (relative to a coordinate system unconnected with the body). Consequently, a rigid body rotating about a fixed point has only *three degrees of freedom* left ; the individual particles can then only move in *spherical surfaces* of which the centre is the fixed point, and whose radii are equal to the invariable distances from that point. The turning of a rigid body about a fixed point is hence also termed a *spherical rotation* of the body.

If *two points* of the rigid body are *fixed*, the individual particles can now only describe *circular paths about one and the same axis*, which is determined by the line joining the two points. We then speak of a *rotation* of the rigid body

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[2] The special case in which n is less than four may be excluded from the following considerations, as being devoid of physical importance.

about a fixed axis. There are now six more equations of condition, expressing the constancy of the coordinates of the two fixed points, to be added to the $(3n - 6)$ arising from the rigidity of the body. Of these six, however, only five are mutually independent, since there already exists one relation between the six coordinates of the two fixed points, which arises from the rigidity of the connection between them. Hence, a body rotating about a fixed axis has only *one degree of freedom.*

If, in a rotation of this kind, the perpendicular dropped on the axis of rotation from a particle turns with an angular velocity $d\varphi/dt$, and if we erect a vector **w** whose direction is that of the axis of rotation, whose magnitude is $d\varphi/dt$, and whose sense is such that, when viewed from its extremity, the rotation of the body appears anti-clockwise, it follows from the considerations of § 12 that the *linear velocity of rotation* of the particle will be given by

(2) $\mathbf{v} = [\mathbf{wr}]$.

In this **r** denotes the radius-vector drawn from any given point on the axis of rotation to the particle in question. Conversely, if the motions of all the particles composing the rigid body can be described by equations of the form of eqn. (2), in all of which equations **w** has the same value, we may conclude that, as long as **w** remains unaltered, the body is rotating about a *fixed axis.* That is to say, none of the particles varies in its distance from a straight line passing through the point from which the radius-vectors **r** are drawn and having the direction of the vector **w**.

When in a rigid body, which can only rotate about a fixed axis, a point lying without that axis is also fixed, the number of degrees of freedom is thereby reduced to *zero,*[3] and the body is unable to move at all.

If, now, the *coordinates* of all the particles ξ_h, η_h, ζ_h ($h = 1$ to n), relative to the previously considered co-ordinate system rigidly connected with the body, are *constant quantities*, as has already been demonstrated, the following expressions must also represent quantities which are *invariable with respect to the time* :—

$$\frac{\sum m_h \xi_h}{\sum m_h}, \quad \frac{\sum m_h \eta_h}{\sum m_h}, \quad \frac{\sum m_h \zeta_h}{\sum m_h}.$$

These expressions, however, signify nothing else than the

[3] Of the additional three equations of condition which then arise, two are already given by the rigidity of the connections between this third point and the other two.

coordinates of the *centre of mass* relative to the fixed coordinate system. Consequently, the distances of the centre of mass from all the individual particles composing the rigid body must be invariable, so that the centre of mass behaves as though it were rigidly connected with the individual particles. We can now construct any number of points whose coordinates with respect to the fixed coordinate system under consideration remain constant, and these we call *points rigidly connected* with the body. They are by no means necessarily associated with masses, but are rather purely geometrical points which are, in general, *in motion* with respect to a system lying without the body. A straight line determined by two such points is termed an *axis rigidly connected* with the body. We can obviously choose any one such point as origin of a coordinate system, and any such rigidly connected axis as an axis of the system, relative to which the coordinates of all the particles of the rigid body will be invariable with respect to the time.

It has already been shown, in connection with *d'Alembert's Principle*, that an *equation of condition* which restricts the freedom of motion of a system of particles can be replaced by one or more *imaginary accessory forces*. Although the magnitudes of the latter naturally depend on the magnitudes and directions of the forces actually exerted, yet we can always recognize the general nature of the imaginary accessory forces which are equivalent to the equations of the form of eqn. (1), by means of the method of *undetermined multipliers*. For, by differentiating eqn. (1) and then substituting the variation symbol δ for the differential symbol d, we have

$$(3) \begin{cases} (x_h - x_k) \, \delta x_h + (y_h - y_k) \, \delta y_h + (z_h - z_k) \, \delta z_h \\ + (x_k - x_h) \, \delta x_k + (y_k - y_h) \, \delta y_k + (z_k - z_h) \, \delta z_k = 0. \end{cases}$$

Multiplying this equation by the factor λ, to which no definite value is as yet attached, and then dividing the left-hand side into two parts, we see that the rigid connection between the hth and the kth particle gives rise to two equally large, but oppositely directed, accessory forces. One of these forces acts upon the hth and the other upon the kth particle, and their components are given by

$$(4) \begin{cases} \lambda \, (x_h - x_k), \quad \lambda \, (y_h - y_k), \quad \lambda \, (z_h - z_k) \\ \text{and} \\ \lambda \, (x_k - x_h), \quad \lambda \, (y_k - y_h), \quad \lambda \, (z_k - z_h) \end{cases}$$

respectively. Since, therefore, the components of these accessory forces exhibit the same mutual relations as the projections upon the coordinate axes of the line joining the

two particles, the *direction* of these *two opposite accessory forces* must be that of the *connecting line* ; *i.e.*, they must be so-called *central forces*.

Hence, in virtue of the earlier considerations of §§ 19 and 20, we can treat a rigid body as a system of *free* particles, upon which there act, besides the external forces actually exerted, imaginary *internal* accessory forces which are central forces, and which obey Newton's Third Law of Motion. It follows that we can apply to a rigid body not only the Principle of the Conservation of the Centre of Mass, but also that of the *Conservation of the Total Angular Momentum*. If we denote the total momentum of the rigid body by $\mathbf{G}$, its total angular momentum referred to a given point by $\mathbf{U}$, and the acting forces by $\mathbf{K}$, the following relations must therefore be satisfied for every rigid body :—

$$(5) \qquad\qquad \frac{d\mathbf{G}}{dt} = \sum \mathbf{K},$$

and

$$(6) \qquad\qquad \frac{d\mathbf{U}}{dt} = \sum [\mathbf{rK}],$$

where $\mathbf{r}$ is the radius-vector drawn from the reference point to the point of application of the force in question.

The Law of the Conservation of Energy is unimportant for the mechanics of rigid bodies, inasmuch as the internal potential of a rigid body must be invariable, owing to the constancy of the distances within it. Hence, the Law of the Conservation of Energy, in its application to rigid bodies, expresses no more than in its application to a single particle. The *kinetic energy* of a rigid body, upon which no external forces act, is thus an invariable quantity.

§ 24. Translation and Rotation of a Rigid Body.

Let O' be a point rigidly connected to a solid body, and let us suppose it to be the origin of a coordinate system which is also rigidly connected with the body. The velocity of the point O' will then represent the *velocity of translation* of the coordinate system, which we shall call $\mathbf{v}_t'$. Let the coordinate system rotate with an *angular velocity* $\mathbf{w}'$.

Let O'' be the origin of a second coordinate system likewise rigidly connected with the body ; let the translatory velocity of this system, *i.e.*, the velocity of the point O'', be $\mathbf{v}_t''$, and its angular velocity $\mathbf{w}''$.

Finally, let us consider an arbitrary point P of the rigid

body, and denote the directed intervals $O'P$, $O''P$ and $O'O''$ by $\mathbf{r'}$, $\mathbf{r''}$ and $\mathbf{d}$ respectively. Then (Fig. 20) :

(1) $$\mathbf{r'} = \mathbf{d} + \mathbf{r''},$$

and, by § 13 eqn. (13),

(2) $$\mathbf{v}_t'' = \mathbf{v}_t' + [\mathbf{w'd}].$$

The velocity of the point P, denoted by $\mathbf{v}$, is thus given either by

(3) $$\mathbf{v} = \mathbf{v}_t' + [\mathbf{w'r'}],$$

or by

(4) $$\mathbf{v} = \mathbf{v}_t'' + [\mathbf{w''r''}].$$

Equating the right-hand sides of eqns. (3) and (4), and taking eqns. (1) and (2) into account, we find

(5) $$[\mathbf{w'r''}] = [\mathbf{w''r''}].$$

The equality of the two vector products in eqn. (5) must be satisfied for any arbitrary direction of $\mathbf{r''}$, while, on the other hand, $\mathbf{w'}$ and $\mathbf{w''}$ are quite independent of $\mathbf{r''}$.[1] Hence we must have

(6) $$\mathbf{w''} = \mathbf{w'}.$$

FIG. 20.

Thus the *angular velocity* with which a coordinate system, rigidly connected with the body, rotates at any given moment is completely *independent* of the position of the origin of this system. At any one moment the angular velocity for all possible coordinate systems, which are rigidly connected with a solid body, has *one and the same value*, which may be denoted by $\mathbf{w}$.

We will now suppose the translatory velocity $\mathbf{v}_t$ to be resolved into two components, one of which ($\mathbf{v}_w$) has the direction of the vector $\mathbf{w}$, while the other ($\mathbf{v}_n$) is perpendicular to that vector (Fig. 21). We then have

(7) $$\begin{cases} \mathbf{v}_t' = \mathbf{v}_w' + \mathbf{v}_n', \\ \mathbf{v}_t'' = \mathbf{v}_w'' + \mathbf{v}_n''. \end{cases}$$

---

[1] Because two vector products [AB] and [AC] are equal, we naturally cannot in general conclude that B = C, but only that B and C lie in one plane with A, and that the parallelograms formed by A and B, and by A and C, are equal in area—for a given direction of A, B and C can yet differ from one another. On the other hand, we readily recognize the impossibility of drawing two differently directed vectors B and C from one point, in such a way that they form parallelograms of the same area and sense of description with *any arbitrary* vector A drawn from the same point. This is only possible if B = C.
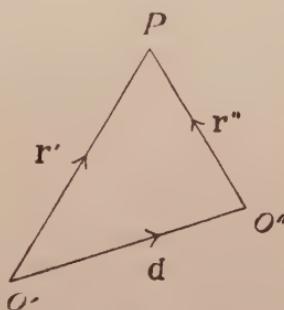
By eqn. (2), the vector which is equal to the vector difference $(\mathbf{v}_t'' - \mathbf{v}_t')$ stands at right angles to the direction of $\mathbf{w}$. The w-component of this vector difference therefore vanishes, and this is only possible if
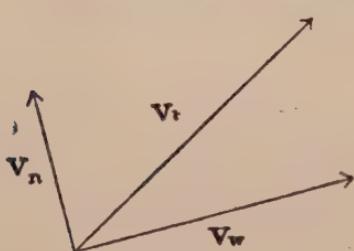
$$(8) \qquad \mathbf{v}_w'' = \mathbf{v}_w';$$

whence, by eqn. (2),

$$(9) \qquad [\mathbf{wd}] = \mathbf{v}_n'' - \mathbf{v}_n'.$$

Hitherto we have assigned no definite value to the vector $\mathbf{d}$, but we will now give it a particular value $\mathbf{d}^*$ such that it stands at right angles to $\mathbf{w}$, and that, furthermore,

$$(10) \qquad [\mathbf{wd}^*] = -\mathbf{v}_n'.$$

To do this we erect the vector $\mathbf{d}^*$ at the point O' perpendicular to the directions of $\mathbf{v}_n'$ and $\mathbf{w}$, giving it a length equal to v_n'/w and a sense such that the rotation serving to carry the vector $\mathbf{w}$ round into the direction of the vector $\mathbf{d}^*$, by the shortest way, appears clockwise when seen from the extremity of the vector $\mathbf{v}_n'$. (In Fig. 21 the vector $\mathbf{d}^*$ will be directed perpendicularly forwards from the plane of the figure.)

We will denote by O^* the extremity of the vector $\mathbf{d}^*$ thus erected at the point O'. If we call the translatory velocity of this point $\mathbf{v}_t^*$, then the component $\mathbf{v}_n^*$ vanishes, according to eqns. (9) and (10); consequently, for the point O^* the translatory velocity has the *same direction* as the vector of the angular velocity. Moreover, it is easy to see that the same property is common to all points which lie in a straight line drawn parallel to $\mathbf{w}$ from the point O^*. For, if $\mathbf{v}_t'''$ be the translatory velocity for an arbitrary point P''' along this line, and $\mathbf{a}$ be the directed line O^*P''', we have, by eqn. (2),

$$(11) \qquad \mathbf{v}_t''' = \mathbf{v}_t^* - [\mathbf{wa}],$$

and the vector product vanishes, $\mathbf{a}$ and $\mathbf{w}$ being similarly directed.

If we choose any point along this line as a point of reference, we can regard the instantaneous motion of the rigid body as being compounded of a translation in the direction of this line, and an instantaneous rotation of the whole body about it, whereby all particles thus describe elements of circular paths about this instantaneous axis. Since a motion consisting of a progressive movement along a straight line

FIG. 21.

combined with a simultaneous circular movement about this line as axis is characteristic of a *screw*, any infinitesimal change in position of a rigid body may be regarded as an *elemental screw motion* about an instantaneous screw axis. This theorem, which forms a starting-point for more advanced and important investigations on the theory of the motion of rigid bodies, is known as *Chasles's Theorem*.

To return to our earlier considerations, let $\mathbf{v}_t'$ be the instantaneous translatory velocity of any given fixed coordinate system whose origin is O', and let $\mathbf{w}$ be the instantaneous angular velocity of the rigid body. Then it follows from § 13 eqn. (13) that the velocity of any particle is given by

$$\mathbf{v} = \mathbf{v}_t' + [\mathbf{w}\mathbf{r}'] ;$$

and we find as values for the *momentum* ($\mathbf{G}$) of the body and for its *angular momentum* ($\mathbf{U}'$), referred to the point O',

(12) $$\mathbf{G} = \sum m\mathbf{v} = \mathbf{v}_t' \sum m + [\mathbf{w} . \sum m\mathbf{r}'],$$

and

(13) $$\mathbf{U}' = \sum m\,[\mathbf{r}'\mathbf{v}] = -[\mathbf{v}_t' . \sum m\mathbf{r}'] + \sum m\,[\mathbf{r}'\,[\mathbf{w}\mathbf{r}']\,].$$

It will be seen that the expressions for $\mathbf{G}$ and $\mathbf{U}'$ assume an especially *simple* form, if we identify the arbitrary reference point O' with the *centre of gravity* of the rigid body; for it is a property of the centre of gravity that $\sum m\mathbf{r}'$ vanishes for it.[2] Hence, calling the velocity of the centre of gravity simply $\mathbf{v}_0$, the radius-vectors drawn from it to the individual particles of the body $\mathbf{r}_0$, and the angular momentum referred to it $\mathbf{U}_0$, we have

(14) $$\mathbf{G} = \mathbf{v}_0 \sum m,$$

and

(15) $$\mathbf{U}_0 = \sum m\,[\mathbf{r}_0\,[\mathbf{w}\mathbf{r}_0]\,].$$

Now the velocity of any single particle is given by

(16) $$\mathbf{v} = \mathbf{v}_0 + [\mathbf{w}\mathbf{r}_0],$$

so that the motion of the rigid body is completely determined as soon as the two vectors $\mathbf{v}_0$ and $\mathbf{w}$ are known. When the acting forces are given as functions of the time, as regards both magnitude and direction, the first of the two vectors can at once be found by means of § 23 eqn. (5) :

(17) $$\sum \mathbf{K} = \frac{d\mathbf{v}_0}{dt} \sum m.$$

[2] For, by § 15 eqn. (7), we have for any given reference point
$$\sum m\mathbf{r} = \mathbf{r}^* \sum m,$$
where $\mathbf{r}^*$ is the directed line drawn from the reference point to the centre of gravity : $\mathbf{r}^*$ vanishes, however, when the reference point coincides with the centre of gravity.

As regards its *translation*, a rigid body therefore behaves just like a single particle ; for it acts as though its whole mass were *concentrated* at the centre of gravity, and as though the whole of the forces, which in reality operate at the various points of the body, were operating at that centre with the same magnitude and direction. Thus the problem of the determination of $\mathbf{v}_0$ appears solved by eqn. (17).[3]

On the other hand we have, by § 23 eqn. (6),

(18) $$\frac{d\mathbf{U}_0}{dt} = \sum [\mathbf{r}_0 \mathbf{K}].$$

From this and eqn. (15) we can calculate the quantity $\mathbf{w}$, when the forces at work are given as functions of the time as regards both magnitude and direction, and also as regards the position of their points of application (relative to the centre of gravity) : in other words, when the moments of the acting forces with respect to the centre of gravity are given. Since the quantities $\mathbf{w}$ and $\mathbf{v}_0$ are quite independent of each other, it follows from eqns. (17) and (18) that $\sum \mathbf{K}$ can vanish without, however, $\sum [\mathbf{rK}]$ also vanishing : in the next section we shall deal more fully with this point.

§ 25.   The Composition of the Forces acting on a Rigid Body.

The simplest representation of a system of forces in which the vector sum of the individual forces vanishes, although the total *moment* of the system relative to any given point differs in value from zero, is that shown in Fig. 22. It consists of two *equally large* and *parallel*, but *oppositely directed forces*, which act on two
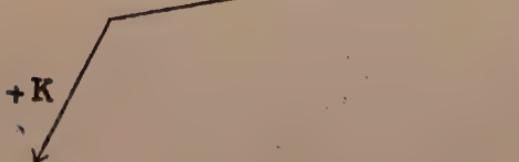

Fig. 22.

points rigidly connected with each other, the line joining the points, however, not coinciding with the direction of the forces. Such a system is termed a *couple*—an expression due to *Poinsot*.

[3] For a given initial value of $\mathbf{v}_0$.

Denoting the one force by $+ \mathbf{K}$, and the other by $- \mathbf{K}$, the vector sum of the two separate forces naturally always vanishes. On the other hand, their *moment* with respect to any *arbitrary* point is, in general, *different from zero*. We shall denote the directed lines drawn from this point to the points of application of the positive and the negative forces by $\mathbf{r}_1$ and $\mathbf{r}_2$, and the directed line drawn from the point of application of the positive force to that of the negative by $\mathbf{a}$. Then

$$\mathbf{r}_2 = \mathbf{r}_1 + \mathbf{a},$$

and thus the moment of the couple with respect to this point is given by

$$\mathbf{M} = [\mathbf{r}_1 \, \mathbf{K}] + [\mathbf{r}_2 \, (- \mathbf{K})] = - [\mathbf{aK}],$$

or,

(1) $$\mathbf{M} = [\mathbf{Ka}].$$

The moment vector $\mathbf{M}$ therefore stands at right angles to the plane formed by the two forces, and its magnitude is equal to the product of the magnitude of either force and the *perpendicular distance* between the parallel directions of the forces. As can be seen from Fig. 22, its sense will be such that the rotation produced by the couple appears anticlockwise when viewed from the extremity of the vector.

Since the vectorial sum of the two separate forces vanishes, a couple can *only* cause a *rotation*, but *not a translation* ; this rotation is completely determined by § 24 eqn. (18) for a given initial value of $\mathbf{U}_0$, as soon as $\mathbf{M}$ is known as a constant or as a function of the time. It at once follows from eqn. (1) that the effect of a couple is entirely unaltered by a *displacement* of the couple parallel to itself, the distance between the directions of the two forces remaining the same : the displacement may be made either within the plane of the couple or parallel to it. All that is essential is that the plane of the couple should maintain a constant direction, and that the product of the absolute amount of the forces and the perpendicular distance between their directions should remain unaltered.

We also see from eqn. (1) that the moment of a couple vanishes when the directions of the two opposite forces coincide (Fig. 23). Such a couple can occasion neither a translation (in accordance with what was stated above) nor a rotation (as we have just shown), so that it has *no effect at all* upon a rigid body.

We can at once draw an important conclusion from this fact. If a force $\mathbf{K}_1$ acts on a point P, we can always apply at any other point P', which lies in the line of action

of $\mathbf{K_1}$, two forces of the same size as $\mathbf{K_1}$, one of which ($\mathbf{K_2}$) has the same direction as $\mathbf{K_1}$, while the other ($\mathbf{K_3}$) has the opposite direction (Fig. 24). Now the forces $\mathbf{K_2}$ and $\mathbf{K_3}$ annul each other, because they are equal and opposite, and act on the same point ; hence the system of the three forces $\mathbf{K_1}$, $\mathbf{K_2}$, $\mathbf{K_3}$ must be equivalent to the single force $\mathbf{K_1}$. On the other hand, in accordance with what was stated above, the forces $\mathbf{K_1}$ and $\mathbf{K_3}$ also annul each other entirely (being a couple whose connecting line coincides with the direction of the forces), so that the force $\mathbf{K_2}$, acting at the point P', is completely equivalent to the equally large and similarly directed force $\mathbf{K_1}$, acting at the point P. We can therefore *displace* the *point* at which a force acts upon a rigid body *in the direction of this force* just as we please, without in any way altering the effects of the latter.

Moreover, in general, when a force $\mathbf{K}$ is acting at a point A, we can always apply two equally large forces at any other entirely arbitrary point B, also rigidly connected with the body, one of these forces being similarly, and the other oppositely directed to $\mathbf{K}$. Since this system of three forces must be equivalent to the single force acting at the point A, we can always *replace* the action of a single force $\mathbf{K}$ at a point A by the united action of an equally large and similarly directed force at any other point B, and a *couple* whose moment is equal to the vector product of $\mathbf{K}$ and the directed line drawn from A to B.

In this manner we can always *combine* the total forces acting on a rigid body into one *resultant single force* $\mathbf{R}$, equal to $\Sigma \, \mathbf{K_h}$, which we may suppose to act at any point O', and a *resultant couple* whose moment is given by

(2) $\mathbf{M'} = [\mathbf{r_h'} \, \mathbf{K_s}],$

FIG. 23.   FIG. 24.

where r_h' denotes the directed lines drawn from O' to the points of application of the forces.

The size and magnitude of the moment vector M' naturally depend on the choice of the reference point O'. If we change over from O' to another point O'', and denote the connecting line $O'O''$ by d, we have

$$r_h' = r_h'' + d.$$

Hence the relation existing between the moment M', with respect to the point O', and the moment M'', with respect to the point O'', follows from eqn. (2) :—

(3) $M'' = M' - [dR]$,

where R is again the resultant of all the individual forces, its point of action being displaced from O' to O''. It follows that, in the case of a rigid body whose centre of gravity is at rest (or in uniform motion), the magnitude and direction of the resultant couple are completely *independent* of the choice of reference point.

By following exactly the same process as was carried out in § 24 with regard to the translatory velocity, we can show that a straight line can always be determined such that, if any point along it be chosen as a reference point, the resultant moment and the resultant single force have the *same direction*. For this purpose we resolve the moment M' into two components, one of which has the same direction as R, while the other is at right angles to the latter : we may call the two components M_r' and M_n'. The vector $[dR]$, which, by eqn. (3), is equal to the geometrical difference of M' and M'', must always stand at right angles to R, in accordance with the definition of a vector product. Consequently, the component of the moment which lies in the direction of R will be unaffected by the transition from one point of reference to another, so that M_r must possess the same value for all reference points. It is only the component M_n that will be altered by a change in reference point

Hence, by a proper choice of d, we can make $[dR]$ equal to M_n'. This we do by constructing a straight line at the point O' at right angles to M_n' and M_r', making its length equal to M_n'/R, and giving it such a sense that the rotation which would serve to carry d into the direction of R, by the shortest way, appears anti-clockwise when viewed from its extremity. We then have

$$[dR] = M_n' ;$$

and thus, by eqn. (3),

$$M'' = M' - M_n' = M_r'.$$

It follows that we can always compound the whole of the forces acting on a rigid body into a resultant single force, together with a couple whose axis of turning coincides with the direction of the resultant single force. Since the component $\mathbf{M}_r$ has the same value for all reference points, that moment for which the component $\mathbf{M}_n$ vanishes completely must represent a *minimum* value as regards its magnitude.[1]

If both the resultant moment and the resultant single force $\mathbf{R}$ *vanish* for a certain reference point, then the former must naturally vanish for *any* reference point whatsoever, seeing that the geometrical difference between the moments, relative to two different points, is equal to the vector product of $\mathbf{R}$ and the line joining the points. Therefore the necessary and, moreover, sufficient condition that the whole of the forces acting on a rigid body should be *in equilibrium* consists in the vanishing of both the resultant single force, and the resultant moment relative to any *arbitrary* point.

§ 26. Tensors.

If $\mathbf{C}$ and $\mathbf{D}$ are any two vectors, it is always possible, with the use of a definite coordinate system, to derive *nine quantities* by multiplying the *components of the two vectors together in pairs*, *i.e.*, by putting

$$(1) \quad \begin{cases} E_{xx} = C_x D_x, & E_{xy} = C_x D_y, & E_{xz} = C_x D_z, \\ E_{yx} = C_y D_x, & E_{yy} = C_y D_y, & E_{yz} = C_y D_z, \\ E_{zx} = C_z D_x, & E_{zy} = C_z D_y, & E_{zz} = C_z D_z. \end{cases}$$

Since

$$(2) \quad E_{xx} + E_{yy} + E_{zz} = (\mathbf{CD}),$$

i.e., the *scalar product* of the vectors $\mathbf{C}$ and $\mathbf{D}$, the sum on the left-hand side of eqn. (2) must be a *scalar* independent of the coordinate system.

If, on the other hand, we consider the *vector product* of the vectors $\mathbf{C}$ and $\mathbf{D}$, we find by eqns. (1) the relations

$$(3) \quad \begin{cases} E_{yz} - E_{zy} = [\mathbf{CD}]_x; \ E_{zx} - E_{xz} = [\mathbf{CD}]_y; \\ E_{xy} - E_{yx} = [\mathbf{CD}]_z. \end{cases}$$

Hence, by forming the expression $(E_{yz} - E_{zy})$ and the two analogous ones, we again obtain the components of a *vector*.

Finally, multiplying the nine quantities E_{xx}, *etc.*, by the

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[1] For the magnitude of the moment is given by the square root of the sum of the squares of the two components.

components of any given vector **B**, according to the following scheme, we obtain

(4) $E_{xx}B_x + E_{xy}B_y + E_{xz}B_z = C_x$ (**BD**), *etc.*

As the scalar product of the vectors **B** and **D** is a scalar, the left-hand sides of eqn. (4) and of the two analogous expressions again represent the components of a *vector*.

Moreover, when the coordinate system is changed, the quantities E_{xx}, *etc.*, can be *transformed* by the rules holding for the transformation of vector components [§ 9 eqn. (3)]. We have

(5) $\begin{cases} E_{x'x'} = C_{x'}D_{x'} = (\alpha_1 C_x + \beta_1 C_y + \gamma_1 C_z) \\ \quad \times (\alpha_1 D_x + \beta_1 D_y + \gamma_1 D_z), \end{cases}$

where α_1, β_1, γ_1, *etc.*, are the cosines of the angles between the old and the new coordinate axes [as in § 9 eqn. (1)]. Bearing in mind eqns. (1), we can also write eqn. (5) in the form

(6) $\begin{cases} E_{x'x'} = \alpha_1{}^2 E_{xx} + \beta_1{}^2 E_{yy} + \gamma_1{}^2 E_{zz} + \alpha_1 \beta_1 (E_{xy} + E_{yx}) \\ \quad + \beta_1 \gamma_1 (E_{yz} + E_{zy}) + \gamma_1 \alpha_1 (E_{zx} + E_{xz}). \end{cases}$

In the same way we find

(7) $E_{x'y'} = (\alpha_1 C_x + \beta_1 C_y + \gamma_1 C_z)(\alpha_2 D_x + \beta_2 D_y + \gamma_2 D_z),$

or,

(8) $\begin{cases} E_{x'y'} = \alpha_1 \alpha_2 E_{xx} + \beta_1 \beta_2 E_{yy} + \gamma_1 \gamma_2 E_{zz} + \alpha_1 \beta_2 E_{xy} \\ + \alpha_2 \beta_1 E_{yx} + \beta_1 \gamma_2 E_{yz} + \beta_2 \gamma_1 E_{zy} + \gamma_1 \alpha_2 E_{zx} + \gamma_2 \alpha_1 E_{xz}. \end{cases}$

If, now, with reference to a definite coordinate system, there are given *nine quantities*

(9) $\begin{cases} k_{xx}, & k_{xy}, & k_{xz}, \\ k_{yx}, & k_{yy}, & k_{yz}, \\ k_{zx}, & k_{zy}, & k_{zz}, \end{cases}$

we *define* these nine quantities as *components of a tensor* when, on a change of the coordinate system, they *are transformed in the same way as the products formed by multiplying the components of two vectors together in pairs.* The quantities in which the same index occurs twice, *viz.*, k_{xx}, k_{yy}, k_{zz}, are termed *tensor components of the first kind*; the other six are called *tensor components of the second kind*.

It follows from the above definition that the possibility of representing tensor components as the products of the components of two vectors is in no way necessary; it is sufficient if they can be *transformed* in the same way as are

the products of the components of two vectors. Where this is the case, we see at once by eqn. (2) that

(10) $\qquad k_{x'x'} + k_{y'y'} + k_{z'z'} = k_{xx} + k_{yy} + k_{zz}:$

the *sum of the tensor components of the first kind* represents a *scalar independent of the coordinate system.*[1] On the other hand, it follows from eqn. (3) that we can *derive from every tensor a vector* independent of the coordinate system, whose components are given by the three expressions

(11) $\qquad k_{yz} - k_{zy}, \quad k_{zx} - k_{xz}, \quad k_{xy} - k_{yx}.$

Finally, we can always obtain a new vector (**A**) by forming relations corresponding to those of eqn. (4) between the components of a tensor and the components of a vector (**B**): thus

(12) $\qquad \begin{cases} k_{xx}B_x + k_{xy}B_y + k_{xz}B_z = A_x, \\ k_{yx}B_x + k_{yy}B_y + k_{yz}B_z = A_y, \\ k_{zx}B_x + k_{zy}B_y + k_{zz}B_z = A_z. \end{cases}$

The vector **A** is then termed a *linear vector function* of the vector **B**, for its components are linear functions of the components of **B**.

Suppose, now, that k_{xx}, etc., are the components of one tensor, and p_{xx}, etc., the components of another: then, *adding* together the corresponding components of the two tensors, we obtain nine new quantities:

(13) $\qquad \begin{cases} q_{xx} = k_{xx} + p_{xx}, \\ q_{xy} = k_{xy} + p_{xy}, \text{ etc.} \end{cases}$

We see immediately from eqns. (6) and (8) that the quantities q_{xx}, q_{xy}, etc., will also be transformed exactly as are the quantities E_{xx}, E_{xy}, etc. Thus, by the *addition of analogous tensor components*, we again obtain the *components of a tensor*.

On the other hand, we can also derive new quantities from the components k_{xx}, etc., of a given tensor, by leaving unaltered the components of the second kind, and increasing each of those of the first kind by one and the same scalar quantity S independent of the coordinate system. The new quantities are then given by the relations

(14) $\quad t_{xx} = k_{xx} + S,\ t_{yy} = k_{yy} + S,\ t_{zz} = k_{zz} + S,\ t_{xy} = k_{xy}, \text{ etc.}$

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[1] We can, naturally, also prove this by forming eqn. (6) for $E_{y'y'}$ and $E_{z'z'}$, adding the three equations together, and substituting the values given by § 9 eqns. (12) and (13) in the result.

Taking into consideration the nine quantities

(15) $\qquad t_{xx} - S,\ t_{yy} - S,\ t_{zz} - S,\ t_{xy},\ t_{yx},\ \text{etc.},\ 20603$

it is obvious from eqn. (14) that they must be transformed in accordance with eqns. (6) and (8), since, of course, k_{xx}, *etc.*, are tensor components. Hence we find

(16) $\begin{cases} t_{x'x'} - S = \alpha_1{}^2\,(t_{xx} - S) + \beta_1{}^2\,(t_{yy} - S) + \gamma_1{}^2\,(t_{zz} - S) \\ \qquad + \alpha_1\beta_1(t_{xy} + t_{yx}) + \beta_1\gamma_1\,(t_{yz} + t_{zy}) + \gamma_1\alpha_1\,(t_{zx} + t_{xz}). \end{cases}$

Now, by § 9 eqn. (9),

(17) $\qquad\qquad S\,(\alpha_1{}^2 + \beta_1{}^2 + \gamma_1{}^2) = S,$

so that eqn. (16) becomes identical with eqn. (6), if the letter t be substituted for the letter E.

If, on the other hand, we insert the nine quantities given by the scheme (15) in eqn. (8), we have

(18) $\begin{cases} t_{x'y'} = \alpha_1\alpha_2\,(t_{xx} - S) + \beta_1\beta_2\,(t_{yy} - S) \\ \qquad\qquad\qquad + \gamma_1\gamma_2\,(t_{zz} - S) + \ \ldots\ ., \end{cases}$

the last six terms (not written) being identical with the last six terms in eqn. (8), if the letter t be substituted for the letter E. By § 9 eqn. (10), however,

(19) $\qquad\qquad \alpha_1\alpha_2 + \beta_1\beta_2 + \gamma_1\gamma_2 = 0,$

so that eqn. (18) becomes identical with eqn. (8), the letter E being replaced by the letter t.

Thus the quantities t_{xx}, t_{xy}, *etc.*, defined by eqn. (14) again represent the components of a tensor. We can therefore derive from any one tensor another tensor, by *increasing the components of the first kind by one and the same scalar*, without altering the components of the second kind.

If, in particular, the *components of the second kind* in any tensor are *equal in pairs*, the tensor is termed *symmetrical*. Hence, for a symmetrical tensor,

(20) $\qquad\quad k_{xy} = k_{yx},\quad k_{yz} = k_{zy},\quad k_{zx} = k_{xz}.$

It may readily be shown that a *symmetrical* tensor can be derived from any given tensor. According to eqns. (6) and (8)

(21) $\quad k_{x'x'} = \alpha_1{}^2\,k_{xx} + \ \ldots\ . + \alpha_1\beta_1\,(k_{xy} + k_{yx}) + \ \ldots\ .,$

and

(22) $\quad k_{x'y'} = \alpha_1\alpha_2\,k_{xx} + \ \ldots\ . + \alpha_1\beta_2\,k_{xy} + \alpha_2\beta_1\,k_{yx} + \ \ldots\ .,$

whereas

(23) $\quad k_{y'x'} = \alpha_2\alpha_1\,k_{xx} + \ \ldots\ . + \alpha_2\beta_1\,k_{xy} + \alpha_1\beta_2\,k_{yx} + \ \ldots$

Let us now derive nine new quantities by means of the relations

(24)
$$\begin{cases} l_{xx} = k_{xx}, \quad l_{yy} = k_{yy}, \quad l_{zz} = k_{zz}, \\ l_{xy} = l_{yx} = \dfrac{1}{2}\,(k_{xy} + k_{yx}), \\ l_{yz} = l_{zy} = \dfrac{1}{2}\,(k_{yz} + k_{zy}), \\ l_{zx} = l_{xz} = \dfrac{1}{2}\,(k_{zx} + k_{xz}). \end{cases}$$

Then

(25) $k_{xy} + k_{yx} = l_{xy} + l_{yx},$

and eqn. (21) can be written in the form

(26) $l_{x'x'} = \alpha_1{}^2\, l_{xx} + \ldots + \alpha_1 \beta_1\,(l_{xy} + l_{yx}) + \ldots$

Again, adding together eqns. (22) and (23), and dividing by two, we may write the result in accordance with eqn. (24)

(27) $l_{x'y'} = \alpha_1 \alpha_2\, l_{xx} + \ldots + \alpha_1 \beta_2\, l_{xy} + \alpha_2 \beta_1\, l_{yx} + \ldots$

Comparing eqns. (26) and (27) with eqns. (21) and (22), we see that the new quantities introduced by eqn. (24) again represent the components of a tensor, and that a symmetrical tensor. Hence, we can *always derive a symmetrical tensor from any given tensor* by means of the relations expressed in eqn. (24). If the original tensor is already symmetrical, it naturally follows from eqn. (24) that the derived tensor is identical with it.

§ 27. The Tensor Ellipsoid.

Let us consider a quite definite coordinate system, relative to which the components of a tensor are k_{xx}, k_{yy}, *etc.*; and let us suppose any straight line to be drawn from the origin and making angles with the axes whose cosines are α, β, γ. As before, we will denote by $k_{x'x'}$ the value assumed by the x-x-component of the tensor when this line is taken as a new x-axis. Then we have by § 26 eqn. (6), omitting the index 1,

(1)
$$\begin{cases} k_{x'x'} = \alpha^2\, k_{xx} + \beta^2\, k_{yy} + \gamma^2\, k_{zz} + \alpha\beta\,(k_{xy} + k_{yx}) \\ \qquad\quad + \beta\gamma\,(k_{yz} + k_{zy}) + \gamma\alpha\,(k_{zx} + k_{xz}). \end{cases}$$

Let us now mark off along this line from the origin an interval, the *square* of whose *length* is the *reciprocal* of the quantity $k_{x'x'}$; and let the coordinates of the *extremity of this interval* be ξ, η, ζ. Since the cosines of the angles in-

cluded between this line-interval and the three coordinate axes are α, β, γ, we have, denoting the length of the segment by l,

(2) $\xi = \alpha\, l, \qquad \eta = \beta\, l, \qquad \zeta = \gamma\, l.$

Again, since l^2 is the reciprocal of k_{xx},

(3) $\alpha^2 = k_{x'x'}\, \xi^2, \qquad \beta^2 = k_{x'x'}\, \eta^2, \qquad \gamma^2 = k_{x'x'}\, \zeta^2.$

Hence, dividing eqn. (1) throughout by $k_{x'x'}$, we find

(4) $\begin{cases} 1 = \xi^2\, k_{xx} + \eta^2\, k_{yy} + \zeta^2\, k_{zz} + \xi\,\eta\,(k_{xy} + k_{yx}) \\ \qquad + \eta\,\zeta\,(k_{yz} + k_{zy}) + \zeta\,\xi\,(k_{zx} + k_{xz}). \end{cases}$

We will now suppose a *bundle of rays* to be drawn *through the origin*, and along each of these rays a length to be marked off from the origin, the square of each such length being the reciprocal of the value assumed by the component $k_{x'x'}$ when the ray in question is taken as x'-axis. The *surface* representing the *locus of the extremities of the lines thus drawn* is then given by eqn. (4), which equation represents a *surface of the second degree*. Where, in particular, the tensor is of such a nature that its *components of the first kind can never become negative*—and we shall only consider this case in what follows—the surface of the second degree can only be an *ellipsoid*.[1] The latter is completely independent of the coordinate system used, and it forms the *graphical representation of a tensor quantity*, just as well as a directed interval does that of a vector. The surface described by eqn. (4) is therefore termed a *tensor ellipsoid*.

The three axes of the ellipsoid determine three mutually *perpendicular* directions which are called the *principal axes of the tensor*. When a tensor differs from place to place, thereby forming a *tensor field*, the orientation of the three principal axes will also, in general, vary continuously from place to place. If the tensor ellipsoid be, in particular, a *spheroid*, only one principal axis has a definite direction, that of each of the other two being indefinite ; in this case, any straight line which is perpendicular to the one definite direction can be regarded as a principal axis. In the special case that the tensor ellipsoid is a *sphere*, any straight line whatsoever may represent a principal axis.

The values assumed by the three tensor components of the first kind, relative to a coordinate system which is formed by the three principal axes, are called the *three principal*

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[1] Otherwise the surface might be a hyperboloid. The reader who is not conversant with analytical solid geometry should think of the analogous relations in analytical plane geometry.

values of the tensor. In the following paragraphs we shall denote them simply by k_1, k_2, k_3. When use is made of a coordinate system formed by the three principal axes, the equation of the ellipsoid obviously becomes *purely quadratic* ; and, in the case of a *symmetrical tensor*, this is only possible [according to eqn. (4)] if the *tensor components of the second kind vanish.*

Hence, in an arbitrary coordinate system, the components of a *symmetrical* tensor can be expressed in terms of the principal values by means of the following relations, which can at once be deduced from § 26 eqns. (6) and (8) :—

$$(5) \qquad \begin{cases} k_{xx} = \alpha_1{}^2 k_1 + \beta_1{}^2 k_2 + \gamma_1{}^2 k_3 , \\ k_{xy} = \alpha_1 \alpha_2 k_1 + \beta_1 \beta_2 k_2 + \gamma_1 \gamma_2 k_3 , \text{ etc.} \end{cases}$$

In these expressions [in accordance with § 9 eqn. (1)], α_1, β_1, γ_1 are the cosines of the angles included between the x-axis and the three principal axes : α_2, β_2, γ_2 are likewise the cosines of the angles between the y-axis and the three principal axes ; and so forth.

If a vector **A** is a *symmetrical vector function* of another vector **B**, and if the components of the two vectors relative to the principal axes of the corresponding tensor are equal respectively to A_1, A_2, A_3 and B_1, B_2, B_3, we have, by § 26 eqn. (12), the simple relations

$$(6) \qquad A_1 = k_1 B_1 , \qquad A_2 = k_2 B_2 , \qquad A_3 = k_3 B_3 .$$

§ 28. Moment of Inertia.

If we denote by **r** the radius-vector drawn to any arbitrary point of a rigid body from a point which is itself at rest (*e.g.*, from a point on a fixed axis of rotation), the velocity of the moving point is given by

$$(1) \qquad \mathbf{v} = [\mathbf{wr}].$$

It follows from § 7 eqn. (3) that the total *angular momentum* relative to the point at rest is

$$\mathbf{U} = \sum m \, [\mathbf{rv}],$$

or, by eqn. (1),

$$(2) \qquad \mathbf{U} = \sum m \big[\mathbf{r} \, [\mathbf{wr}] \big].$$

But, by a frequently used formula of vectorial algebra [§ 3 eqn. (24)],

$$(3) \qquad \big[\mathbf{r} \, [\mathbf{wr}] \big] = \mathbf{w} \, (\mathbf{rr}) - \mathbf{r} \, (\mathbf{rw}) ;$$

or, in accordance with the formula for the scalar product of two vectors, and remembering that the components of $\mathbf{r}$ are the coordinates x, y, z of the particle,

(4) $\big[\mathbf{r}\,[\mathbf{w}\mathbf{r}]\big] = \mathbf{w}r^2 - \mathbf{r}\,(xw_x + yw_y + zw_z).$

Hence, writing eqn. (2) analytically instead of vectorially, we have

(5) $\begin{cases} U_x = w_x \sum m\,(r^2 - x^2) - w_y \sum mxy - w_z \sum mxz, \\ U_y = -\,w_x \sum myx + w_y \sum m(r^2 - y^2) - w_z \sum myz, \\ U_z = -\,w_x \sum mzx - w_y \sum mzy + w_z \sum m\,(r^2 - z^2). \end{cases}$

We see at once that the *nine summands* occurring in this triplet of equations are the *components of a tensor*. For, if we form the nine expressions mx^2, mxy, etc., for a particle, they will be transformed in the same way, naturally, as the products of the components of two vectors, seeing that x, y, z are actually the components of the vector $\mathbf{r}$. Therefore these nine expressions undoubtedly represent the components of a tensor, and, moreover, of a symmetrical tensor. Now, if we take the sum of these tensor components over the whole of the particles of the body, it follows from § 26 eqn. (13) that we must again obtain the components of a tensor, and it makes no difference if we reverse the sign throughout. Finally, moreover, we also obtain the components of a tensor if, in accordance with § 26 eqn. (14), we leave the tensor components of the second kind unaltered but add to those of the first kind the scalar

(6) $S = \sum mr^2.$

In this way, however, we arrive at the nine summands of eqn. (5), and so these are proved to be the components of a *symmetrical tensor*. This tensor is known as the *tensor of the moment of inertia*.

Since

(7) $r^2 = x^2 + y^2 + z^2,$

the tensor components of the first kind can never be zero or negative [by eqn. (5)]. The surface of the second degree which represents the tensor is therefore an ellipsoid, and this is termed the *ellipsoid of inertia*. Its construction is due to *Poinsot* (1834), who was the first to recognize in this important example the possibility of the graphical representation of a tensor.

It follows from eqns. (5) and (7) that the tensor components of the first kind, referred to an arbitrary

coordinate system, which may be denoted by I_{xx}, *etc.*, are given by

$$(8) \quad \begin{cases} I_{xx} = \sum m\,(y^2 + z^2), \ I_{yy} = \sum m\,(y^2 + x^2), \\ \quad\quad I_{zz} = \sum m\,(x^2 + y^2). \end{cases}$$

Now $(y^2 + z^2)$ is the square of the distance of a point from the x-axis ; and so we can find the three tensor components of the first kind by multiplying the masses of the individual particles by the squares of their distances from the three coordinate axes, and then taking the sum over all the particles. The three components of the first kind are consequently also called the *moments of inertia about the three coordinate axes*.

We will now consider any arbitrary *axis* passing through the *centre of gravity* of the rigid body ; let I be the moment of inertia about this axis—*i.e.*, I represents the value assumed by I_{xx} when this axis is taken as the x-axis. Let there be a second axis parallel to the one under consideration, and let the moment of inertia about it be I'. We will denote the perpendicular dropped from any given point of the body on to the axis passing through the centre of gravity by $\mathbf{p}$, and that dropped on to the second axis by $\mathbf{p}'$. Let $\mathbf{d}$ be the distance between the two axes, as measured from the axis passing through the centre of gravity to the second axis. Then

$$(9) \quad\quad\quad\quad \mathbf{p}' = \mathbf{p} + \mathbf{d},$$

and hence

$$(10) \quad\quad I' = I + d^2 \sum m + 2d \sum mp \cos \gamma,$$

where γ is the angle between the vectors $\mathbf{p}$ and $\mathbf{d}$.

Let us now lay a coordinate system through the centre of gravity in such a way that its ξ-axis coincides with the direction of $\mathbf{d}$; then

$$(11) \quad\quad\quad \sum mp \cos \gamma = \sum m\xi.$$

This sum, however, is equal to the product of the total mass and the ξ-coordinate of the centre of gravity, and consequently vanishes when the origin coincides with the latter. Hence we have simply

$$(12) \quad\quad\quad\quad I' = I + d^2 \sum m.$$

This important relation is known as *Steiner's Theorem*, after its discoverer. The moment of inertia about any given axis is thus equal to that about a parallel axis passing through the centre of gravity, increased by the product of the total mass of the body and the square of the distance of the axis in question from the centre of gravity. Consequently, an

axis passing through the centre of gravity is distinguished from a manifold of parallel axes by the fact that the moment of inertia about it is a minimum.

The axes of the ellipsoid of inertia constructed for the centre of gravity are known briefly as the *principal axes of inertia* of the body, and the moments of inertia relative to them are termed the three *principal moments of inertia.* Denoting the latter by I_1, I_2, I_3, let us consider any axis passing through the centre of gravity, and let α, β, γ be the cosines of the angles included between this axis and the principal axes of inertia. Then, according to § 27 eqn. (5), the moment of inertia about this axis is given by

(13) $$I = \alpha^2 I_1 + \beta^2 I_2 + \gamma^2 I_3 ;$$

and the moment of inertia about any other parallel axis can be obtained by means of Steiner's theorem.

If we make use of a coordinate system formed by the principal axes of inertia, the connection between angular momentum and angular velocity follows from § 27 eqn. (6) in the simple form

(14) $$U_1 = w_1 I_1, \quad U_2 = w_2 I_2, \quad U_3 = w_3 I_3.$$

In conclusion, we shall give a few simple *examples* of the *calculation of moments of inertia.* As a first example we will take the case of a *rectangular parallelopiped*, the lengths of whose sides are a, b, c, and whose density (ϱ) is constant. We shall take the directions of the three edges as coordinate axes, and the centre of gravity of the parallelopiped as origin ; and we wish to find the moments of inertia about the three axes. We have

$$I_{xx} = \sum m \, (y^2 + z^2) = \varrho \int_{-\frac{a}{2}}^{+\frac{a}{2}} \int_{-\frac{b}{2}}^{+\frac{b}{2}} \int_{-\frac{c}{2}}^{+\frac{c}{2}} (y^2 + z^2) \, dx \, dy \, dz$$

$$= \varrho a \left(\int y^2 \, dy \int dz + \int z^2 \, dz \int dy \right) = \varrho a \, \frac{b^3 c + c^3 b}{12}.$$

Now the product ϱabc represents the total mass of the parallelopiped, so that, calling this m, we find

(15) $$I_{xx} = m \, \frac{b^2 + c^2}{12}, \quad I_{yy} = m \frac{c^2 + a^2}{12}, \quad I_{zz} = m \, \frac{a^2 + b^2}{12}.$$

If two of the sides can be disregarded in comparison with the third side, *e.g.*, the side a, the moment of inertia about an axis perpendicular to the long side will be

(16) $$I_{yy} = I_{zz} = \frac{ma^2}{12},$$

whereas the third moment of inertia can be neglected in comparison with the other two.

As a second example of the calculation of a moment of inertia we will consider a *circular cylinder* of height h and radius a. We see that two co-axial surfaces of the cylinder of radii r and $(r + dr)$ enclose a volume $2\pi h r \, dr$, and that the amount contributed towards the total moment of inertia, about the axis, by the mass of this volume, is obtained by multiplying this expression by ϱ and r^2. Hence we find for the moment of inertia of the whole cylinder about its axis

$$I = 2\pi\varrho h \int_0^a r^3 \, dr = 2\pi\varrho h \, \frac{a^4}{4}.$$

But the total mass (m) of the cylinder is equal to $\pi\varrho a^2 h$; and therefore

$$(17) \qquad I = \frac{ma^2}{2}.$$

As a third and final example, we will calculate the moment of inertia of a *sphere*. To do this we first consider a spherical shell bounded by two spherical surfaces of radii r and $(r + dr)$. It follows from the complete symmetry that the moment of inertia of this shell about any axis passing through the centre must have the same value. Hence, calling this moment of inertia I', we have

$$I' = \sum m (y^2 + z^2) = \sum m (z^2 + x^2) = \sum m (x^2 + y^2).$$

Again,

$$r^2 = x^2 + y^2 + z^2,$$

and consequently

$$3 I' = 2 \sum m r^2 = 2r^2 \sum m.$$

The mass of the spherical shell is $4\pi\varrho r^2 dr$, and therefore

$$(18) \qquad I' = \frac{8}{3} \pi\varrho r^4 \, dr.$$

The moment of inertia of the whole sphere of radius a is accordingly given by

$$(19) \qquad I = \frac{8}{3} \pi\varrho \int_0^a r^4 \, dr = \frac{8}{3} \pi\varrho \, \frac{a^5}{5}.$$

Again, the mass of the sphere, which may be denoted by m, is

$$m = \frac{4}{3} \pi\varrho a^3 \, ;$$

and so we find for the moment of inertia of the sphere the value

$$(20) \qquad\qquad I = \frac{2}{5}\, ma^2.$$

§ 29. Rotation about a Fixed Axis.

Let us first consider the *kinetic energy of rotation* of a rigid body rotating about a fixed axis. If the magnitude of the angular velocity be w, and if p be the perpendicular dropped from any point of the body on to the axis, then the magnitude of the velocity of that point is wp. Hence we find for the kinetic energy of the rotating body

$$(1) \qquad\qquad L = \frac{1}{2}\sum mp^2 w^2 = \frac{1}{2}\, I w^2,$$

where I is the moment of inertia about the axis of rotation. Eqn. (1) is completely analogous to the relation which determines the kinetic energy of a single particle as being half the product of the mass and the square of the velocity. The quantity I therefore plays a precisely similar part, in the mechanics of rigid bodies, to that played by inertial mass in the dynamics of a single particle ; and in this we have the explanation of the term " moment of inertia."

We arrive at another simple expression by forming the scalar product of § 28 eqn. (3) and $\mathbf{w}$: we then have

$$\mathbf{w}\left[\mathbf{r}\left[\mathbf{wr}\right]\right] = w^2\, r^2 - (\mathbf{wr})^2 = w^2\, r^2 \left\{1 - \cos^2(\mathbf{w},\mathbf{r})\right\}$$
$$= w^2\, r^2 \sin^2(\mathbf{w},\mathbf{r}).$$

But $r \sin(\mathbf{w},\mathbf{r})$ is equal to the perpendicular p : hence, by § 28 eqn. (2),

$$(2) \qquad\qquad \mathbf{U}\mathbf{w} = w^2 \sum mp^2 = w^2\, I,$$

or, by eqn. (1),

$$(3) \qquad\qquad L = \frac{1}{2}\, \mathbf{U}\mathbf{w}.$$

During the rotation of a body, the rigid connection between all the particles forming the body and the axis of rotation gives rise to *reaction forces*. Since, in the case of a rotation about a fixed axis, the particles describe circular paths, the reaction force represents the *centrifugal force*. Denoting by $\mathbf{K}^*$ its value for any given particle, and by $\mathbf{p}$ the perpendicular dropped from the particle on to the axis of rotation, we have

$$(4) \qquad\qquad \mathbf{K}^* = -\, mw^2\, \mathbf{p}.$$

Now, in accordance with § 25, we can compound the total centrifugal forces into a resultant single force $\mathbf{R}^*$ and a resultant couple of moment $\mathbf{M}^*$. Then

(5) $$\mathbf{R}^* = -w^2 \sum m\mathbf{p},$$

and

(6) $$\mathbf{M}^* = -w^2 \sum m[\mathbf{rp}],$$

where $\mathbf{r}$ is the radius-vector drawn from the reference point to the particle in question.

We will now make the axis of rotation the x-axis of a coordinate system whose origin coincides with the reference point, so that

$$p_x = 0, \quad p_y = -y, \quad p_z = -z; \quad r_x = x, \quad r_y = y, \quad r_z = z.$$

Hence

(7) $$R_x^* = 0, \quad R_y^* = w^2 \sum my, \quad R_z^* = w^2 \sum mz.$$

Moreover,

$$[\mathbf{rp}]_x = -yz + zy = 0, \quad [\mathbf{rp}]_y = xz, \quad [\mathbf{rp}]_z = -xy;$$

and thus

(8) $$M_x^* = 0, \quad M_y^* = -w^2 \sum mxz, \quad M_z^* = w^2 \sum mxy,$$

or, by § 28 eqn. (5),

(9) $$M_x^* = 0, \quad M_y^* = w^2 I_{xz}, \quad M_z^* = -w^2 I_{xy}.$$

It follows from § 15 eqn. (7) that the resultant single force of all the centrifugal forces vanishes when the *axis of rotation passes through the centre of gravity*. Further, in accordance with eqn. (9), the moment of the resultant couple also vanishes when the axis of rotation coincides with a *principal axis of inertia*.

When this is not the case, the vector $\mathbf{M}^*$ differs from zero and is certainly otherwise directed than the x-axis, *i.e.*, its direction differs from that of the axis of rotation. Although the motion of the particles occurs parallel to the y-z-plane when the axis of rotation is fixed, the component M_y^* tends to *deflect* the motion in the direction of the x-z-plane, and the component M_z^* in that of the x-y-plane. The components of the second kind in the tensor of the moment of inertia are accordingly termed *moments of deviation*.

§ 30. Euler's Equations.

The *equations of motion of a rigid body* assume an especially simple form, when the motion is referred to a *coordinate system* whose *origin is the centre of gravity*, and whose *axes*

are the principal axes of inertia. Denoting by **U** the angular momentum relative to the centre of gravity (or, if the latter be at rest, simply the angular momentum), and by **M** the resultant moment of all the applied forces relative to the centre of gravity, we have, by § 24 eqn. (18),

$$(1) \qquad \mathbf{M} = \frac{d\mathbf{U}}{dt}.$$

Again, denoting by $d*\mathbf{U}/dt$ the time-derivative of the angular momentum with respect to a coordinate system rigidly connected to the body, we have, by § 12 eqn. (17),

$$(2) \qquad \frac{d\mathbf{U}}{dt} = \frac{d*\mathbf{U}}{dt} + [\mathbf{wU}].$$

In accordance with § 12 eqn. (17) we may form an analogous equation by replacing the vector **U** by the vector **w** itself. Since, however, the vector product of a vector with itself vanishes, we find simply

$$(3) \qquad \frac{d\mathbf{w}}{dt} = \frac{d*\mathbf{w}}{dt}.$$

Transforming eqn. (1) from the vectorial into the analytical form, and taking eqn. (2) into account, we obtain, on choosing as coordinate axes the principal axes of inertia.

$$(4) \qquad M_1 = \frac{d*U_1}{dt} + w_2 U_3 - w_3 U_2.$$

Again, it follows from § 28 eqn. (14) as well as from eqn. (3) that

$$(5) \qquad \frac{d*U_1}{dt} = I_1 \frac{dw_1}{dt}.$$

Hence, bearing in mind § 28 eqn. (14), we obtain the following triplet of equations from eqn. (4) :—

$$(6) \qquad \begin{cases} M_1 = I_1 \frac{dw_1}{dt} - w_2 w_3 (I_2 - I_3), \\[2mm] M_2 = I_2 \frac{dw_2}{dt} - w_3 w_1 (I_3 - I_1), \\[2mm] M_3 = I_3 \frac{dw_3}{dt} - w_1 w_2 (I_1 - I_2). \end{cases}$$

These important *equations of motion of a rigid body* were first established by *Euler* in 1765.

By making use of the above equations (which, moreover, form the basis of the theory of gyroscopic motion), it

becomes an easy matter to answer the question as to whether and when a body can *rotate continuously and uniformly about a fixed axis, in the absence of external forces.* If such is to be the case, the left-hand sides of the three eqns. (6) must vanish, because of the absence of external forces, and likewise the first terms on the right-hand sides, seeing that the vector **w** is required to be constant. The three final terms in the equations must therefore also vanish; and this is only possible for a quite arbitrary body whose three principal moments of inertia must be assumed to differ from each other, if

(7) $w_2\,w_3 = w_3\,w_1 = w_1\,w_2 = 0.$

Eqns. (7) are satisfied either when all three components are zero, in which case no rotation at all can occur, and this is here excluded from consideration, or when two of the three components of the angular velocity vanish. This, however, can only be the case when the direction of the vector **w** is permanently that of one of the three principal axes of inertia. The *principal axes of inertia* therefore at the same time represent the so-called *free axes* of a rigid body, which are defined as being those axes about which the rigid body can rotate uniformly in the absence of external forces.

§ 31. The Compound Pendulum.

As a simple example of the motion of a rigid body we will consider the *compound pendulum.* By this we mean a rigid

body which is free to turn about a horizontal axis, and which executes an oscillatory motion under the influence of its own weight. The *moment* of the effective force is given by the product of the weight of the pendulum and the distance between the horizontal axis of rotation and a vertical line passing through the centre of gravity. This distance is equal to $a \sin \varphi$, where a is the distance between the centre of gravity and the horizontal axis, and φ the angle made with the vertical by the line joining the axis and the centre of gravity (Fig. 25). Hence the magnitude of the restoring moment of the force of gravity is given by

FIG. 25.

(1) $M = mga \sin \varphi,$

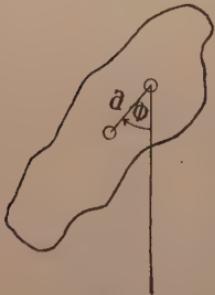where m is the mass of the pendulum and g the acceleration due to gravity.

Now, in this instance, the direction of the vector **M** must by definition be that of the horizontal axis of rotation. Its sense will be such that the rotation, produced by the force of gravity, appears anti-clockwise when viewed from the extremity of the vector ; thus in Fig. 25 the vector **M** will be directed forwards from the plane of the figure.

The differential coefficient $d\varphi/dt$ represents the *angular velocity*. The vector of the angular velocity likewise has the direction of the axis of rotation ; and, by definition, its sense is determined by the condition that a rotation for which the angle φ increases must appear anti-clockwise as viewed from its extremity : in Fig. 25 the vector of the angular velocity will thus be directed backwards from the plane of the figure. The same applies to the vector **U** of the *angular momentum*, seeing that all points of the pendulum carry out a circular motion. Consequently the moment of force and the angular momentum are oppositely directed,[1] so that, by § 24 eqn. (18),

$$(2) \qquad M = -\frac{dU}{dt}.$$

Since the vectors **U** and **w**, in accordance with what has just been stated, are similarly directed, and since the magnitude of the vector **w** is equal to $d\varphi/dt$, we have, by § 29 eqn. (2),

$$(3) \qquad U = I \frac{d\varphi}{dt},$$

where I is the moment of inertia about the horizontal axis of rotation. Hence, combining eqns. (1), (2) and (3),

$$(4) \qquad mga \sin \varphi = -I \frac{d^2\varphi}{dt^2}.$$

If we only take into account *small displacements* for which $\sin \varphi$ may be put equal to φ, eqn. (4) assumes the form

$$(5) \qquad \frac{d^2\varphi}{dt^2} = -\frac{mga}{I}\,\varphi.$$

On the other hand, in § 6 we have derived the following differential equation for the case of a pendulum consisting of only one particle (and termed a *mathematical* or *simple pendulum*) :—

$$\frac{d^2\varphi}{dt^2} = -\frac{g}{l}\,\varphi.$$

---

[1] This likewise holds good when the pendulum is displaced towards the right instead of the left, as may easily be seen from Fig. 25.

We see that a compound pendulum oscillates in the same manner as a simple pendulum of length

$$(6) \qquad\qquad l^* = \frac{I}{ma}.$$

The quantity l^* is called the *length of the equivalent simple pendulum*. If a perpendicular be dropped from the centre of gravity on to the axis of rotation, and a length l^* be cut off along it from the latter, we thereby find the so-called *centre of oscillation*. In spite of its rigid connection with the other particles composing the pendulum, it oscillates just as if it alone were existent, and as though it were attached to the axis of rotation by means of a weightless rod. The period of a *compound* pendulum is equal to that of a simple pendulum of length l^*, and is hence given by

$$(7) \qquad\qquad \tau = 2\pi \sqrt{\frac{l^*}{g}}.$$

The notion of the centre of oscillation is due to *Huygens*, who was the first to develop the theory of the compound pendulum, in 1673.

Now, by *Steiner's theorem*,

$$(8) \qquad\qquad I = I^* + ma^2,$$

where I^* is the moment of inertia about an axis parallel to that of rotation and passing through the centre of gravity. Hence

$$(9) \qquad\qquad l^* = \frac{I^*}{ma} + a.$$

We see from this equation that the length of the equivalent simple pendulum must be greater than the distance of the centre of gravity from the axis of rotation. Thus, in the equilibrium position of the pendulum, the centre of oscillation always lies *below* the centre of gravity.[2]

By means of eqn. (9) we can also calculate the quantity a, if the period or, what amounts to the same thing, the

---

[2] Taking, for example, a sphere of radius r, we find [by § 28 eqn. (20)] the length of the equivalent simple pendulum to be

$$l^* = \frac{2r^2}{5a} + a.$$

The distance of the centre of oscillation below the centre of the sphere is therefore, $2r^2/5a$. Thus, if $a < 2r/5$, the centre of oscillation lies outside the sphere.

length of the equivalent simple pendulum be given. Multiplying eqn. (9) by a, we find

$$(10) \qquad a^2 - al^* = -\frac{I^*}{m}.$$

This quadratic equation has *two roots, viz.*,

$$(11) \qquad \left. \begin{array}{c} a_1 \\ a_2 \end{array} \right\} = \frac{l^*}{2} \pm \sqrt{\frac{l^{*2}}{4} - \frac{I^*}{m}} :$$

hence

$$(12) \qquad a_1 + a_2 = l^*.$$

If, therefore, a pendulum whose centre of suspension is at a distance a from the centre of gravity has a period τ, and if we find that the period is the same when the centre of suspension is changed to a distance a' (the new axis of rotation being parallel to the old), we have by eqns. (7) and (12) the relation

$$(13) \qquad a + a' = \frac{\tau^2 g}{4 \pi^2}.$$

In this way we may determine the *acceleration due to gravity* by measuring the quantities a, a', and τ, for a given position of the centre of gravity.[3]

---

[3] The position of the centre of gravity of bodies in the form of thin sheets can easily be determined empirically. If the body be suspended at any point, the centre of gravity must lie in the vertical line passing through the point of suspension, when the body is in equilibrium. If, therefore, the body be successively suspended from several points, and the vertical lines be drawn each time, all these lines must intersect in one point, which then represents the centre of gravity of the laminal body.

GENERAL THEORY OF VECTOR FIELDS

§ 32. Differential Operations in Vector Analysis.

IF a coordinate system be situated in a *vector field*, then the *three components of a vector* **A** may be regarded as *functions of the coordinates*, so that we can form *nine partial derivatives* according to the following scheme :—

$$
(1) \quad
\begin{cases}
\dfrac{\partial A_x}{\partial x} & \dfrac{\partial A_x}{\partial y} & \dfrac{\partial A_x}{\partial z} \\[2ex]
\dfrac{\partial A_y}{\partial x} & \dfrac{\partial A_y}{\partial y} & \dfrac{\partial A_y}{\partial z} \\[2ex]
\dfrac{\partial A_z}{\partial x} & \dfrac{\partial A_z}{\partial y} & \dfrac{\partial A_z}{\partial z} .
\end{cases}
$$

Let us now investigate the manner in which these nine quantities will be *transformed*, on passing to another coordinate system. For this purpose we will first calculate the expression $\partial A_{x'}/\partial x'$: by § 9 eqn. (3)

$$(2) \qquad A_{x'} = \alpha_1 A_x + \beta_1 A_y + \gamma_1 A_z,$$

and similarly, since x, y, z are the components of a vector which is drawn from the origin to the point under consideration in the field,

$$(3) \qquad x' = \alpha_1 x + \beta_1 y + \gamma_1 z.$$

On the other hand, by § 9 eqn. (5),

$$(4) \qquad x = \alpha_1 x' + \alpha_2 y' + \alpha_3 z'.$$

In these equations α_1, β_1, γ_1 are the cosines of the angles included between the new x-axis and the three old axes, *etc.* —in accordance with § 9 eqn. (1).

Since $A_{x'}$ is a function of the coordinates x, y, z, we have

$$(5) \qquad \frac{\partial A_{x'}}{\partial x'} = \frac{\partial A_{x'}}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial A_{x'}}{\partial y}\frac{\partial y}{\partial x'} + \frac{\partial A_{x'}}{\partial z}\frac{\partial z}{\partial x'}.$$

But, by eqn. (4) and the two analogous expressions for y and z,

$$(6) \qquad \frac{\partial x}{\partial x'} = \alpha_1, \quad \frac{\partial y}{\partial x'} = \beta_1, \quad \frac{\partial z}{\partial x'} = \gamma_1;$$

and again, by eqn. (2),

(7) $$\frac{\partial A_{x'}}{\partial x} = \alpha_1 \frac{\partial A_x}{\partial x} + \beta_1 \frac{\partial A_y}{\partial x} + \gamma_1 \frac{\partial A_z}{\partial x}.$$

Hence

$$
(8) \quad
\begin{cases}
\dfrac{\partial A_{x'}}{\partial x'} = \alpha_1 \alpha_1 \dfrac{\partial A_x}{\partial x} + \alpha_1 \beta_1 \dfrac{\partial A_y}{\partial x} + \alpha_1 \gamma_1 \dfrac{\partial A_z}{\partial x} \\[2mm]
\qquad + \beta_1 \alpha_1 \dfrac{\partial A_x}{\partial y} + \beta_1 \beta_1 \dfrac{\partial A_y}{\partial y} + \beta_1 \gamma_1 \dfrac{\partial A_z}{\partial y} \\[2mm]
\qquad + \gamma_1 \alpha_1 \dfrac{\partial A_x}{\partial z} + \gamma_1 \beta_1 \dfrac{\partial A_y}{\partial z} + \gamma_1 \gamma_1 \dfrac{\partial A_z}{\partial z}.
\end{cases}
$$

In complete analogy to eqn. (5) we have

$$\frac{\partial A_{x'}}{\partial y'} = \frac{\partial A_{x'}}{\partial x}\frac{\partial x}{\partial y'} + \frac{\partial A_{x'}}{\partial y}\frac{\partial y}{\partial y'} + \frac{\partial A_{x'}}{\partial z}\frac{\partial z}{\partial y'},$$

or, by eqn. (6),

(9) $$\frac{\partial A_{x'}}{\partial y'} = \alpha_2 \frac{\partial A_{x'}}{\partial x} + \beta_2 \frac{\partial A_{x'}}{\partial y} + \gamma_2 \frac{\partial A_{x'}}{\partial z}.$$

We therefore obtain the partial derivative $\partial A_{x'}/\partial y'$ by substituting the index 2 for the index 1 in the first factor in all the terms of eqn. (8). A comparison with § 26 eqns. (6) and (8) shows us that the nine partial derivatives are transformed, in point of fact, in like manner to the products of the components of two vectors. Thus, the *nine partial derivatives of the components of a vector with respect to the coordinates* represent the *components of a tensor*.

From this fact we can at once draw three important conclusions. According to § 26 eqn. (10), the *sum of the tensor components of the first kind* must represent a *scalar* independent of the coordinate system ; in our case, using an expression the significance of which will be evident later (§ 46), we define this scalar as the *divergence of the vector*, and indicate it by the symbol *div* **A**. That is to say,

(10) $$\operatorname{div} \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}.$$

On the other hand, it follows from § 26 eqn. (11) that, if we take the *differences of the tensor components of the second kind in pairs*, we again obtain the components of a *vector*, which, in our case, we term the *rotation of the vector* [1]—an expression the significance of which will likewise be evident

---

[1] Alternative expressions are the " rotor of **A**," or the " curl of **A**." The last expression is that most commonly used in English works.

later. The symbol used for it is *rot* **A**. Its components relative to an arbitrary coordinate system are determined by the following equations, in which, however, the *sense* of the rotation is first fixed by definition, by the order of succession of the terms on the right-hand side :—[2]

$$
(11) \quad
\begin{cases}
\operatorname{rot}_x \mathbf{A} = \dfrac{\partial A_z}{\partial y} - \dfrac{\partial A_y}{\partial z}, \\[2mm]
\operatorname{rot}_y \mathbf{A} = \dfrac{\partial A_x}{\partial z} - \dfrac{\partial A_z}{\partial x}, \\[2mm]
\operatorname{rot}_z \mathbf{A} = \dfrac{\partial A_y}{\partial x} - \dfrac{\partial A_x}{\partial y}.
\end{cases}
$$

Again, by § 26 eqn. (24), we can derive from the vector a *symmetrical tensor* for every part of the field, and this may be termed the *dilatation of the vector*.[3] Its components are

$$
(12) \quad
\begin{cases}
t_{xx} = \dfrac{\partial A_x}{\partial x}, \quad t_{yy} = \dfrac{\partial A_y}{\partial y}, \quad t_{zz} = \dfrac{\partial A_z}{\partial z}, \\[2mm]
t_{xy} = t_{yx} = \dfrac{1}{2}\left(\dfrac{\partial A_x}{\partial y} + \dfrac{\partial A_y}{\partial x} \right), \\[2mm]
t_{yz} = t_{zy} = \dfrac{1}{2}\left(\dfrac{\partial A_y}{\partial z} + \dfrac{\partial A_z}{\partial y} \right), \\[2mm]
t_{zx} = t_{xz} = \dfrac{1}{2}\left(\dfrac{\partial A_z}{\partial x} + \dfrac{\partial A_x}{\partial z} \right).
\end{cases}
$$

Finally, if we multiply the nine partial derivatives of the vector **A**, with respect to the coordinates, by the components of another vector **B**, in a similar manner to that given in § 26 eqn. (12), we again obtain the components of a *vector* (according to § 26). The latter is known as the *gradient of the vector* **A** *relative to the vector* **B**, and is denoted by the symbol (**B** *grad*) **A**. It follows from the definition of this operation that

$$
(13) \quad
\begin{cases}
(\mathbf{B}\ \mathrm{grad})\ \mathbf{A} = \mathbf{i} \left\{ \dfrac{\partial A_x}{\partial x}\ B_x + \dfrac{\partial A_x}{\partial y}\ B_y + \dfrac{\partial A_x}{\partial z}\ B_z \right\} \\[2mm]
\qquad\qquad + \mathbf{j} \left\{ \dfrac{\partial A_y}{\partial x}\ B_x + \dfrac{\partial A_y}{\partial y}\ B_y + \dfrac{\partial A_y}{\partial z}\ B_z \right\} \\[2mm]
\qquad\qquad + \mathbf{k} \left\{ \dfrac{\partial A_z}{\partial x}\ B_x + \dfrac{\partial A_z}{\partial y}\ B_y + \dfrac{\partial A_z}{\partial z}\ B_z \right\}
\end{cases}
$$

[2] *Vide* § 46 eqn. (13).
[3] This term is otherwise not in general use, and it is only introduced for the considerations of this book.

or, by § 10 eqn. (5),

(14) $\quad \begin{cases} (\mathbf{B} \text{ grad}) \, \mathbf{A} = \mathbf{i} \, (\mathbf{B} \text{ grad } A_x) + \mathbf{j} \, (\mathbf{B} \text{ grad } A_y) \\ \qquad\qquad + \mathbf{k} \, (\mathbf{B} \text{ grad } A_z). \end{cases}$

We shall be able to understand the differential operation represented by this equation, if we consider a point O in the vector field at which the vector has the value $\mathbf{A_0}$, and cut off around it a small *region* within which is situated a point P, where the value of the vector is $\mathbf{A}$. Let $\mathbf{r}$ be the directed line OP. Then, by § 10 eqn. (10),

$$A_x = A_{x^0} + \mathbf{r} \text{ grad } A_x ,$$

or, in accordance with the rules for scalar products

(15) $\qquad A_x = A_{x^0} + r_x \, \frac{\partial A_x}{\partial x} + r_y \, \frac{\partial A_x}{\partial y} + r_z \, \frac{\partial A_x}{\partial z}.$

Hence, as a comparison with eqn. (13) shows,

(16) $\qquad\qquad \mathbf{A} = \mathbf{A_0} + (\mathbf{r} \text{ grad}) \, \mathbf{A}.$

Eqn. (15) can readily be given a different form, which will be important for later considerations, by introducing the components of the dilatation of $\mathbf{A}$; we put

(17) $\begin{cases} A_x = A_{x^0} + \dfrac{\partial A_x}{\partial x} \, r_x + \dfrac{1}{2} \left(\dfrac{\partial A_x}{\partial y} + \dfrac{\partial A_y}{\partial x} \right) r_y \\[3mm] \qquad\quad + \dfrac{1}{2} \left(\dfrac{\partial A_x}{\partial z} + \dfrac{\partial A_z}{\partial x} \right) r_z + A_x''. \end{cases}$

Comparing eqns. (17) and (15), we see that the quantity A_x'' (whose value we shall first determine) is given by

$$A_x'' = \frac{1}{2} \left(\frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial x} \right) r_y + \frac{1}{2} \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) r_z ,$$

or,

$$2 \, A_x'' = - \text{ rot}_z \, \mathbf{A} \cdot r_y + \text{rot}_y \, \mathbf{A} \cdot r_z .$$

Thus the quantity A_x'', together with the analogously formed quantities A_y'' and A_z'', represent the components of a vector $\mathbf{A}''$, which, in its turn, is equal to half the vector product of the rotation of $\mathbf{A}$ and the vector $\mathbf{r}$; *i.e.*,

(18) $\qquad\qquad \mathbf{A}'' = \frac{1}{2} \, [\text{rot } \mathbf{A}, \mathbf{r}].$

We may therefore put

(19) $\qquad\qquad \mathbf{A} = \mathbf{A_0} + \frac{1}{2} \, [\text{rot } \mathbf{A}, \mathbf{r}] + \mathbf{A}',$

where the vector $\mathbf{A}'$ is a linear function of the vector $\mathbf{r}$ and of the dilatation of $\mathbf{A}$.

We will now pass from the simple to the so-called combined differential operations of vector analysis. Since the gradient of a scalar which forms a field itself represents a vector, we may further form its divergence and rotation. The *divergence of the gradient* of a scalar is also known as the *Laplace derivative* of the scalar, and the symbol $\nabla^2 S$ is used to denote it. We have

$$\text{div grad } S = \nabla^2 S = \frac{\partial}{\partial x}(\text{grad}_x S) + \frac{\partial}{\partial y}(\text{grad}_y S)$$
$$+ \frac{\partial}{\partial z}(\text{grad}_z S),$$

or, by § 10 eqn. (5),

(20) $$\text{div grad } S = \nabla^2 S = \frac{\partial^2 S}{\partial x^2} + \frac{\partial^2 S}{\partial y^2} + \frac{\partial^2 S}{\partial z^2}.$$

If **A** be any arbitrary vector, we have

$$\nabla^2 A_x = \frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_x}{\partial y^2} + \frac{\partial^2 A_x}{\partial z^2}.$$

Forming the analogous equations for A_y and A_z, multiplying the three respectively by **i**, **j**, **k**, and then adding, we obtain the following expression on the right-hand side :—

(21) $$\frac{\partial^2 \mathbf{A}}{\partial x^2} + \frac{\partial^2 \mathbf{A}}{\partial y^2} + \frac{\partial^2 \mathbf{A}}{\partial z^2}.$$

Since the differentiation of a vector once or twice with respect to a scalar again gives a vector, expression (21) represents the sum of three vectors, and is therefore itself a vector which we may denote by $\nabla^2 \mathbf{A}$. Thus

(22) $$\nabla^2 \mathbf{A} = \mathbf{i} \, \nabla^2 A_x + \mathbf{j} \, \nabla^2 A_y + \mathbf{k} \, \nabla^2 A_z.$$

For the *rotation of a gradient* we find

$$\text{rot}_x(\text{grad } S) = \frac{\partial}{\partial y}(\text{grad}_z S) - \frac{\partial}{\partial z}(\text{grad}_y S)$$
$$= \frac{\partial}{\partial y}\left(\frac{\partial S}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial S}{\partial y}\right) = 0.$$

As analogous relations hold for the y- and the z- components of the rotation of the gradient, we have

(23) $$\text{rot grad } S = 0.$$

Furthermore, we can of course form the divergence and the rotation of the rotation of a vector. If we first work out the *divergence of the rotation*, we find

$$\frac{\partial}{\partial x}(\text{rot}_x \mathbf{A}) = \frac{\partial}{\partial x}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right):$$

and hence, by adding the two equations obtained from cyclic interchange,

$$\text{div rot } \mathbf{A} = \left(\frac{\partial^2 A_z}{\partial x \, \partial y} - \frac{\partial^2 A_y}{\partial x \, \partial z} \right) + \left(\frac{\partial^2 A_x}{\partial y \, \partial z} - \frac{\partial^2 A_z}{\partial y \, \partial x} \right)$$
$$+ \left(\frac{\partial^2 A_y}{\partial z \, \partial x} - \frac{\partial^2 A_x}{\partial z \, \partial y} \right).$$

Thus we have

(24) $$\text{div rot } \mathbf{A} = 0 .$$

On the other hand,

$$\text{rot}_x (\text{rot } \mathbf{A}) = \frac{\partial}{\partial y} (\text{rot}_z \mathbf{A}) - \frac{\partial}{\partial z} (\text{rot}_y \mathbf{A})$$
$$= \frac{\partial}{\partial y} \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) - \frac{\partial}{\partial z} \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right)$$
$$= \frac{\partial^2 A_y}{\partial x \, \partial y} - \frac{\partial^2 A_x}{\partial y^2} - \frac{\partial^2 A_x}{\partial z^2} + \frac{\partial^2 A_z}{\partial x \, \partial z} ,$$

or,

$$\text{rot}_x (\text{rot } \mathbf{A}) = \frac{\partial}{\partial x} (\text{div } \mathbf{A}) - \nabla^2 A_x .$$

Consequently,

(25) $$\text{rot rot } \mathbf{A} = \text{grad div } \mathbf{A} - \nabla^2 \mathbf{A} .$$

We will now calculate the divergence and the rotation of the product of a scalar and a vector, on the one hand, and on the other, of the vector product of two vectors, since the results are of importance for later considerations. First of all, then, we will calculate div $(S\mathbf{A})$, whereby both the scalar S and the vector $\mathbf{A}$ are considered as being space-functions. We have

$$\text{div } (S\mathbf{A}) = \frac{\partial A_x}{\partial x} S + \frac{\partial A_y}{\partial y} S + \frac{\partial A_z}{\partial z} S$$
$$+ A_x \frac{\partial S}{\partial x} + A_y \frac{\partial S}{\partial y} + A_z \frac{\partial S}{\partial z} ,$$

or,

(26) $$\text{div } (S\mathbf{A}) = S \text{ div } \mathbf{A} + \mathbf{A} \cdot \text{grad } S .$$

Similarly, we find

$$\text{rot}_x (S\mathbf{A}) = \frac{\partial A_z}{\partial y} S - \frac{\partial A_y}{\partial z} S + A_z \frac{\partial S}{\partial y} - A_y \frac{\partial S}{\partial z},$$

or

(27) $$\text{rot } (S\mathbf{A}) = S \text{ rot } \mathbf{A} + [\text{grad } S, \mathbf{A}$$

Again, in order to find the divergence of a vector product, we start from the equation

$$[\mathbf{AB}]_x = A_y\, B_z - A_z\, B_y \,.$$

We have

$$\frac{\partial}{\partial x}\,[\mathbf{AB}]_x = \frac{\partial A_y}{\partial x}\, B_z + \frac{\partial B_z}{\partial x}\, A_y - \frac{\partial A_z}{\partial x}\, B_y - \frac{\partial B_y}{\partial x}\, A_z \,,$$

$$\frac{\partial}{\partial y}\,[\mathbf{AB}]_y = \frac{\partial A_z}{\partial y}\, B_x + \frac{\partial B_x}{\partial y}\, A_z - \frac{\partial A_x}{\partial y}\, B_z - \frac{\partial B_z}{\partial y}\, A_x \,,$$

$$\frac{\partial}{\partial z}\,[\mathbf{AB}]_z = \frac{\partial A_x}{\partial z}\, B_y + \frac{\partial B_y}{\partial z}\, A_x - \frac{\partial A_y}{\partial z}\, B_x - \frac{\partial B_x}{\partial z}\, A_y \,.$$

Consequently,

$$\text{div}\,[\mathbf{AB}] = - A_x\,\text{rot}_x\,\mathbf{B} - A_y\,\text{rot}_y\,\mathbf{B} - A_z\,\text{rot}_z\,\mathbf{B}$$
$$+ B_x\,\text{rot}_x\,\mathbf{A} + B_y\,\text{rot}_y\,\mathbf{A} + B_z\,\text{rot}_z\,\mathbf{A} \,.$$

Now we recognize on the right-hand side of this equation the scalar products of $\mathbf{A}$ and rot $\mathbf{B}$, and of $\mathbf{B}$ and rot $\mathbf{A}$ respectively, so that

(28) $$\text{div}\,[\mathbf{AB}] = - \mathbf{A}\,\text{rot}\,\mathbf{B} + \mathbf{B}\,\text{rot}\,\mathbf{A} \,.$$

In like manner we find

$$\text{rot}_x\,[\mathbf{AB}] = \frac{\partial}{\partial y}\,[\mathbf{AB}]_z - \frac{\partial}{\partial z}\,[\mathbf{AB}]_y = \frac{\partial}{\partial y}\,(A_x B_y - A_y B_x)$$
$$+ \frac{\partial}{\partial z}\,(A_z B_x - A_x B_z) \,,$$

or,

$$\text{rot}_x\,[\mathbf{AB}] = A_x\,\text{div}\,\mathbf{B} - A_x\,\frac{\partial B_x}{\partial x} - A_y\,\frac{\partial B_x}{\partial y} - A_z\,\frac{\partial B_x}{\partial z}$$
$$- B_x\,\text{div}\,\mathbf{A} + B_x\,\frac{\partial A_x}{\partial x} + B_y\,\frac{\partial A_x}{\partial y} + B_z\,\frac{\partial A_x}{\partial z}$$

Hence, taking eqn. (13) into account, we have

(29) $$\left\{ \begin{array}{l} \text{rot}\,[\mathbf{AB}] = \mathbf{A}\,\text{div}\,\mathbf{B} - \mathbf{B}\,\text{div}\,\mathbf{A} \\ \quad + (\mathbf{B}\,\text{grad})\,\mathbf{A} - (\mathbf{A}\,\text{grad})\,\mathbf{B}. \end{array} \right.$$

Finally, let us investigate the gradient of the scalar product of two vectors, but only for the special case in which the scalar product is formed from a vector multiplied by itself, as only this case will be required. We have, first,

$$\text{grad}_x\,(\mathbf{AA}) = 2\left\{ A_x\,\frac{\partial A_x}{\partial x} + A_y\,\frac{\partial A_y}{\partial x} + A_z\,\frac{\partial A_z}{\partial x} \right\} \,.$$

By writing this, however, in the form

$$\frac{1}{2} \operatorname{grad}_x (A^2) = A_x \frac{\partial A_x}{\partial x} + A_y \frac{\partial A_x}{\partial y} + A_z \frac{\partial A_x}{\partial z}$$
$$+ A_y \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) + A_z \left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z} \right),$$

we can utilize eqn. (13), and we then find

(30) $$\operatorname{grad} \left(\frac{A^2}{2} \right) = (\mathbf{A} \operatorname{grad}) \, \mathbf{A} + [\mathbf{A}, \operatorname{rot} \mathbf{A}] \, .$$

§ 33. Gauss's Theorem.

Let us consider a surface-element df within a vector field, and let $\mathbf{A}$ be the value of the vector at the place in question; we shall denote by A_n the component of the vector in the direction of the normal to the surface-element, this normal *pointing outwards*. The product $A_n \, df$ is called the *flux of the vector through the surface-element*. The *integral of the normal component of the vector taken over a closed surface* is termed the *total vector flux through that closed surface*.

Now, we may regard A_n as the scalar product of the vector $\mathbf{A}$ and a unit-vector $\mathbf{n}$ pointing in the outward direction of the normal. The components of this unit-vector are $\cos (\mathbf{n}, x)$, $\cos (\mathbf{n}, y)$, $\cos (\mathbf{n}, z)$; and hence [by § 3 eqn. (13)] the total vector flux is

(1) $$\begin{cases} \int A_n \, df = \int A_x \cos (\mathbf{n}, x) \, df + \int A_y \cos (\mathbf{n}, y) \, df \\ \qquad\qquad + \int A_z \cos (\mathbf{n}, z) \, df \, . \end{cases}$$

In order to calculate the flux of the vector we may suppose a coordinate system to be constructed, and the space

Fig. 26.

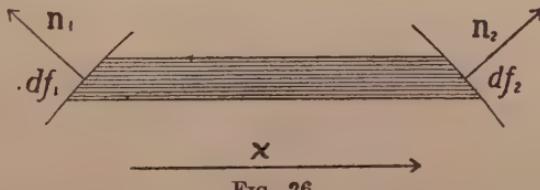bounded by the surface to be cut up into slices parallel to the x-y-plane, these slices being again divided parallel to the x-z-plane. In this way we obtain *rod-shaped elements* lying in the *direction of the x-axis*. Let $\mathbf{n}_1$ and $\mathbf{n}_2$ be the unit-vectors normal to the surfaces bounding the ends of such a rod (Fig. 26), their sense being outwards, and let the areas

of the end-surfaces be df_1 and df_2. Let S_1 and S_2 be the values, in the positions of the two surface-elements, of an *arbitrary*, single-valued, continuous *function (S) of the co-ordinates*. We shall suppose the index 1 to correspond to the position with the smaller x-coordinate, and the index 2 to that with the greater.

We will now calculate the integral

$$\int S \cos (\mathbf{n}, x)\, df\ .$$

It is obvious that we obtain this integral by forming the expression

$$S_1\, df_1 \cos (\mathbf{n}_1, x) + S_2\, df_2 \cos (\mathbf{n}_2, x)$$

for each individual rod, and then taking the sum for the whole of the rods. But

$$df_1 \cos (\mathbf{n}_1, x) = -\, dy\, dz\ ,$$

the *negative* sign being necessary because the outward normal $\mathbf{n}_1$ must always make an *obtuse angle* with the x-axis, when df_1 is the surface-element with the smaller x-coordinate. On the other hand,

$$df_2 \cos (\mathbf{n}_2, x) = +\, dy\, dz\ ,$$

because $\mathbf{n}_2$ clearly makes an *acute angle* with the x-axis.

Thus the part contributed by the rod-shaped element under consideration to the integral for the whole closed surface is equal to

$$(S_2 - S_1)\, dy\, dz\ .$$

Now

(2) $$S_2 - S_1 = \int_{x_1}^{x_2} \frac{\partial S}{\partial x}\, dx\ ,$$

where x_1 and x_2 are the x-coordinates of the two surfaces bounding the ends of the rod. Hence we find for the integral taken over the whole closed surface

(3) $$\int S \cos (\mathbf{n}, x)\, df = \iiint \frac{\partial S}{\partial x}\, dx\, dy\, dz = \int \frac{\partial S}{\partial x}\, d\tau\ ,$$

where $d\tau$ denotes the *volume-element*.

Eqn. (3) holds for any arbitrary, single-valued, continuous function of the coordinates. Since A_x, A_y, A_z are such functions, we can, by means of eqn. (3), transform each of the three integrals on the right-hand side of eqn. (1). We thus find

$$\int A_n\, df = \int \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right) d\tau\ ,$$

or

$$(4) \qquad \int A_n \, df = \int \operatorname{div} \mathbf{A} \, d\tau.$$

The *flux of a vector through a closed surface* is *equal to the integral of the divergence of the vector, taken over the total volume enclosed by the surface*. This important relation is known as *Gauss's Theorem*, after its discoverer.

We can obtain another important theorem from Gauss's theorem if, instead of leaving the vector **A** as an arbitrary quantity, we put it equal to the product of a scalar and the gradient of the scalar. It follows from § 32 eqns. (26) and (20) that

$$(5) \qquad \operatorname{div} (S \operatorname{grad} S) = S \, \nabla^2 S + (\operatorname{grad} S)^2.$$

On the other hand we have, by Gauss's theorem, on integrating over a closed surface,

$$(6) \qquad \int \operatorname{div} (S \operatorname{grad} S) \, d\tau = \int S \operatorname{grad}_n S \, df.$$

When, in particular, the field is of such a nature that the *gradient of the scalar vanishes at its boundaries*, both sides of eqn. (6) become zero, and we find on integrating over the whole field, in accordance with eqn. (5),

$$(7) \qquad \int S \, \nabla^2 S \, d\tau = - \int (\operatorname{grad} S)^2 \, d\tau.$$

This important relation is called *Green's Theorem*, after its discoverer.[1]

§ 34. Vector Lines.

If we consider a vector field at any given moment, we can construct a curve through every point in it in the following way. We proceed from any point in the direction which the vector has at that point to a neighbouring one ; thence in the direction of the vector at that second point ; and so on. In this way we obtain a curve whose *line-element* ($d\mathbf{s}$) has everywhere *the same direction as the vector*, and whose differential equation may hence be written

$$(1) \qquad [\mathbf{A} \, d\mathbf{s}] = 0 \; .$$

A curve of this kind is termed a *vector line*. Two vector lines can obviously *never intersect*, for, were they to do so, the vector would simultaneously have two different direc-

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[1] Other integral theorems besides eqn. (7) can be derived from Gauss's theorem, and these, together with eqn. (7), are collectively known as Green's Theorems.

tions at the point of intersection, and that, of course, is
impossible. A bundle of vector lines passing through a small
surface represents a so-called *vector tube*. When its section
is very small, it is also called a *vector filament*.

Let us now suppose a piece to be cut out of a vector tube
of this kind, the piece being bounded by two surfaces at
right angles to the tube (Fig. 27). Let the vector lines
(whose sense is that of the vector)
enter the tube through the one
end-surface, whose area is q_1; and
let them *pass out* through the other
end-surface, whose area is q_2. Let
A_1 and A_2 be the magnitudes of
the vector in the positions of the
two ends. Then the sides of the
tube and the two ends enclose a
volume, and the integral of the
divergence of the vector taken
over this volume must, by Gauss's
theorem, be equal to the total flux
of the vector through the surfaces
bounding the volume. There is,
however, no vector flux through
the sides of the tube, for the direction of the vector is
everywhere tangential to the sides, which are formed of
vector lines. Since, moreover, the two end-surfaces are at
right angles to the tube, we have
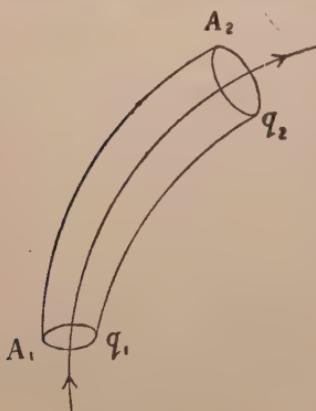
$$(2) \qquad \int \operatorname{div} \mathbf{A}\, d\tau = -A_1 q_1 + A_2 q_2$$

(for the normal pointing outwards from the first end is
opposite in direction to $\mathbf{A}$).

We may draw important conclusions from eqn. (2) for
the special case that the vector $\mathbf{A}$ itself represents the
rotation of another vector. The field of the vector $\mathbf{A}$ is then
termed a *vortex field*, and we accordingly speak of *vortex lines*,
vortex tubes, and *vortex filaments*. According to § 32 eqn. (24),
the *divergence vanishes* throughout a vortex field, and we
thus find from eqn. (2) the simple relation

$$(3) \qquad A_1 q_1 = A_2 q_2 .$$

Consequently, along a vortex tube, the *product of the
magnitude of the vector and the cross-section of the tube* is con-
stant.[1] Conversely, we can find out how the magnitude of

[1] This product is called the *moment* of the vortex tube.

Fig. 27.

the vector varies locally in a vortex field, by dividing up the latter into vortex tubes ; where any given tube contracts, the magnitude of the vector correspondingly increases. The direction of the vector naturally agrees everywhere with the direction of the tubes. We can also see at once from eqn. (3) that *vortex lines can neither begin nor end within the vortex field*, and must therefore represent *closed curves* within the field.

We can thus form a simple graphical representation of a vortex field by constructing vortex tubes ; and, since the Greek word for tube is *solén*, the field of a vector whose divergence vanishes throughout is also known as a *solenoidal* (or *toroidal*) *vector field*. The antithesis of a vortex field is represented by the field of a vector which, in its turn, is the *gradient of a scalar*. A vector field of this kind can be most simply represented by the construction of the *field-levels* of the scalar. In accordance with the definition of a gradient (§ 10), the vector everywhere stands *perpendicular* to the field-levels, whereby its direction is determined. Again, it follows from § 10 eqn. (10) that the relation

$$(4) \qquad\qquad a_{12} = \frac{S_2 - S_1}{A}$$

must be satisfied, where a_{12} is the perpendicular distance between two neighbouring level-surfaces, and A is the magnitude of the vector. Hence, if we suppose the vector field to be cut up by the field-levels into thin slices (so-called *lamellæ*), the *thickness of the lamellæ* must, by eqn. (4), be everywhere *inversely proportional to the magnitude of the vector* (Fig. 28). Since a vector field of this kind can be graphically represented in a simple manner by the construction of lamellæ, the field of a vector which

Fig. 28.

itself forms the gradient of a scalar is termed a *lamellar vector field*.

If, now, we are given an *entirely arbitrary vector field*, which therefore need be neither solenoidal nor lamellar, we are able to *represent graphically* not only the direction of the vector at every point, but, by means of a simple stipulation, also the *magnitude of the vector*. This stipulation simply need require that the *number of vector lines* which pass through any surface-element in the field shall be *equal to the vector flux* through that element, apart from an arbitrary
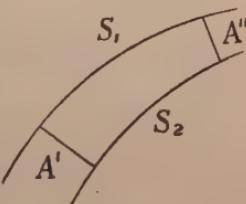
proportional factor which will be the same throughout the field. In particular, however, when the field does not happen to be solenoidal, we must allow *vector lines to begin or end* within the vector tubes. Within a portion of a tube which is bounded as in Fig. 27, just as many new vector lines must arise as is given by the integral of the divergence taken over the enclosed volume, multiplied by the above proportional factor. If this volume-integral be negative, it determines the number of vector lines which end within the portion of the tube considered.

§ 35. Stokes's Theorem.

Since Gauss's theorem renders it readily possible to transform a surface-integral, taken over a closed surface, into an integral taken over the enclosed volume, it naturally occurs to us to investigate whether it be not also possible to *transform a line-integral, taken along a closed curve, into a surface-integral*, taken over the surface enclosed by the curve.

We begin by cutting off a small region round a point O in a vector field, and making it so small that the partial derivatives of the vector components with respect to the coordinates may be regarded as constant within it. With the point O as origin we construct a coordinate system (x, y, z), whose axes correspond to the principal axes of the dilatation tensor at this position. Then, within this region, we have, by § 32 eqn. (19),

$$(1) \qquad \mathbf{A} = \mathbf{A_0} + \frac{1}{2} \left[\text{rot } \mathbf{A}, \mathbf{r} \right] + \mathbf{A}'.$$

Inasmuch as $\mathbf{A}'$ is a linear function of the vector $\mathbf{r}$ (whose components are x, y, z) and of the dilatation tensor, it follows from § 32 eqn. (12) that

$$(2) \qquad A'_x = x \frac{\partial A_x}{\partial x}, \quad A'_y = y \frac{\partial A_y}{\partial y}, \quad A'_z = z \frac{\partial A_z}{\partial z}.$$

We now suppose any *closed plane curve* to be drawn *within the small region*, such that it and all its elements lie completely within this region. Then, according to eqn. (1) the *line-integral of the vector along the curve* is given by

$$(3) \quad \left\{ \begin{array}{l} \displaystyle\int \mathbf{A} \, d\mathbf{s} = \mathbf{A_0} \int d\mathbf{s} + \frac{1}{2} \int \left[\text{rot } \mathbf{A}, \mathbf{r} \right] d\mathbf{s} \\[2ex] \displaystyle + \int A'_x \, dx + \int A'_y \, dy + \int A'_z \, dz. \end{array} \right.$$

But the integral taken along a curve between its beginning and its end is

$$(4) \qquad \int d\mathbf{s} = \mathbf{i}\,(x_2 - x_1) + \mathbf{j}\,(y_2 - y_1) + \mathbf{k}\,(z_2 - z_1),$$

where x_1, y_1, z_1 are the coordinates of the initial point, and x_2, y_2, z_2 are those of the end point. In the case of a closed curve, however, the initial and end points coincide, so that the first term on the right-hand side of eqn. (3) vanishes.

Again, since the partial derivatives of the vector components with respect to the coordinates are to be regarded as constant within the region considered, we have, by eqn. (2),

$$(5) \qquad \int A'_x\, dx = \frac{\partial A_x}{\partial x} \int x\, dx.$$

For a closed curve, however,

$$\int x\, dx = \frac{x_2{}^2}{2} - \frac{x_1{}^2}{2} = 0.$$

Hence the last three integrals in eqn. (3) vanish, and the latter reduces to

$$(6) \qquad \int \mathbf{A}\, d\mathbf{s} = \frac{1}{2} \int [\text{rot } \mathbf{A},\, \mathbf{r}]\, d\mathbf{s}.$$

Now, by a formula of vectorial algebra which has often been used [§ 3 eqn. (23)],

$$(7) \qquad [\text{rot } \mathbf{A},\, \mathbf{r}]\, d\mathbf{s} = [\mathbf{r}\, d\mathbf{s}]\, \text{rot } \mathbf{A}.$$

Seeing that the partial derivatives of the vector components with respect to the coordinates are to be regarded as constant within the region, the same must also hold for the rotation of the vector, so that eqn. (6) may be written in the form

$$(8) \qquad \int \mathbf{A}\, d\mathbf{s} = \text{rot } \mathbf{A} \int \frac{1}{2}\, [\mathbf{r}\, d\mathbf{s}].$$

The magnitude of $\frac{1}{2}\, [\mathbf{r}\, d\mathbf{s}]$, however, is nothing else than the area of a triangle whose vertex is the point O, and whose base is the segment $d\mathbf{s}$ (Fig. 29). The integral on the right-hand side of eqn. (8) is therefore a vector of magnitude equal to the area of the surface enclosed by the curve, of direction perpendicular to the plane of that surface, and of which
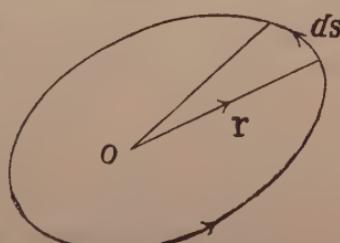

Fig. 29.

the sense is such that the bounding curve appears to be described anti-clockwise as viewed from its extremity.[1]

If we call the unit-vector lying in this direction **n**, and the area of the surface enclosed by the curve f, the integral on the right-hand side of eqn. (8) is equal to $\mathbf{n}f$, and hence, for a *sufficiently small region*,

$$(9) \qquad \int \mathbf{A}\, d\mathbf{s} = \mathbf{n} \operatorname{rot} \mathbf{A}\, f.$$

The scalar product of the rotation and the unit-vector **n**, however, simply represents the *normal component of the rotation* with respect to the surface. Eqn. (9) is all the more exactly fulfilled the smaller the area enclosed by the curve along which the line-integral is to be taken.

If we now turn our attention to any arbitrarily large surface, which is therefore always of finite size and does not require to be a plane surface, we can easily recognize the p o s s i b i l i t y of reducing this more complicated case to the simpler one already solved, by *dividing up* the finite surface into a number of *infinitesimal plane surface - elements* (Fig. 30). On forming the line-integral for all of these elements, we see that it occurs *twice* for all the curve-elements, except

Fɪɢ. 30.

for those which make up the bounding curve, and that its direction is *opposite* in these two cases. Hence all the elements of the line-integral vanish, with the exception of those relating to the *bounding curve*. Since the line-integral over a surface-element is equal to the product of the area of the surface and the normal component of the rotation of the vector for the place in question, we find for the line-integral, taken along the bounding curve, the important relation

$$(10) \qquad \int \mathbf{A}\, d\mathbf{s} = \int \mathbf{n} \operatorname{rot} \mathbf{A}\, df.$$

The *line-integral of a vector along a closed curve* is equal to the

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[1] This follows from the fact that the rotation serving to carry the vector **r** round into the direction of the vector $d\mathbf{s}$, by the shortest way, must appear anti-clockwise when viewed from the extremity of the vector.
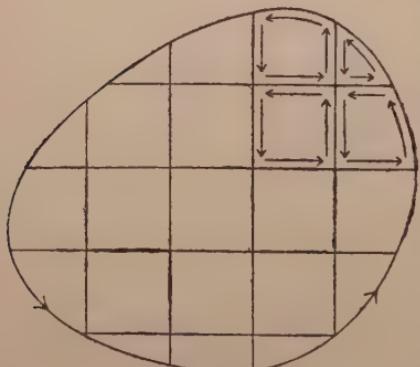
integral of the normal component of its rotation, taken over a surface enclosed by the curve.[2] This relation is known as *Stokes's Theorem* after its discoverer.

This theorem enables us to calculate the integral, taken along a curve, of any scalar dependent upon position, *i.e.*, the integral

$$\int S \, d\mathbf{s},$$

which represents a vector. In order to transform this integral in such a way that Stokes's theorem may be applicable, we make use of an artifice, which consists in forming the scalar product of the integral and a vector **a** which, within the field, is constant. We then have, on applying Stokes's theorem,

$$(11) \qquad \mathbf{a} \int S \, d\mathbf{s} = \int S\mathbf{a} \, d\mathbf{s} = \int \mathbf{n} \operatorname{rot} (S\mathbf{a}) \, df.$$

Now, by § 32 eqn. (27),

$$(12) \qquad \operatorname{rot} (S\mathbf{a}) = S \operatorname{rot} \mathbf{a} - [\mathbf{a} \operatorname{grad} S].$$

The first term on the right-hand side of eqn. (12) vanishes, because **a** has to be independent of the coordinates. Hence

$$(13) \qquad \mathbf{a} \int S \, d\mathbf{s} = - \int \mathbf{n} \, [\mathbf{a} \operatorname{grad} S] \, df.$$

By a formula of vectorial algebra which we have often used [§ 3 eqn. (23)]

$$(14) \qquad \mathbf{n} \, [\mathbf{a} \operatorname{grad} S] = \mathbf{a} \, [\operatorname{grad} S, \mathbf{n}] = - \mathbf{a} \, [\mathbf{n} \operatorname{grad} S].$$

Thus, since the constant vector **a** which is independent of position can be put before the integration symbol, we have

$$(15) \qquad \mathbf{a} \int S \, d\mathbf{s} = \mathbf{a} \int [\mathbf{n} \operatorname{grad} S] \, df.$$

This equation must be satisfied no matter what the magnitude and the direction of the vector **a** may be. From this follows the general validity of the important relation

$$(16) \qquad \int S \, d\mathbf{s} = \int [\mathbf{n} \operatorname{grad} S] \, df.$$

Just as in eqn. (10), the normal to the surface is to be so

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[2] Only the boundary of the surface is of import : its form is entirely beside the point.

constructed that the sense in which the curve is described appears anti-clockwise, when viewed from the extremity of the normal.

§ 36. Tensor Fields and Vector Divergence.

A region over which a tensor is so distributed that the tensor components appear as continuous differentiable functions of the coordinates is termed a *tensor field.* The total number of partial derivatives with respect to the coordinates then amounts to 27.

If, now, we *multiply the components of three vectors together in tripleis*, we likewise obtain 27 quantities following the scheme

$$(1) \qquad F_{xxx} = A_x B_x C_x, \quad F_{xxy} = A_x B_x C_y,$$

etc. Hence

$$(2) \qquad F_{xxx} + F_{xyy} + F_{xzz} = A_x (\mathbf{BC}).$$

When 27 quantities k_{xxx}, k_{xxy}, *etc.*, are given relative to a coordinate system, we define these quantities as being the components of a *tensor of the third order*, if they are transformed in the same way as the triple products of the components of three vectors.[1] It follows from this definition that we can *derive a vector* from the components of a tensor of the third order, by means of the relations

$$(3) \qquad \begin{cases} V_x = k_{xxx} + k_{xyy} + k_{xzz}, \\ V_y = k_{yxx} + k_{yyy} + k_{yzz}, \\ V_z = k_{zxx} + k_{zyy} + k_{zzz}. \end{cases}$$

It clearly follows from the considerations of § 26, without having to repeat the intricate calculations in a far more complicated form, that the 27 partial derivatives of the components of a tensor, with respect to the coordinates, must represent the components of a tensor of the third order. In accordance with eqn. (3), by partial differentiation with respect to the coordinates and corresponding addition we can thus derive from any tensor (t) a vector, which is termed the *vector divergence* of the tensor : the symbol **div** t is used to denote it. (For the sake of distinction, the earlier discussed divergence of a vector is also called the scalar divergence. The symbol for scalar divergence is always written

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[1] Tensors, in the narrower sense of the word, may be regarded as tensors of the second order ; vectors and scalars as tensors of the first and of zero order respectively.

in Roman type, whilst that for vector divergence is written in Clarendon type.) Thus we have

$$(4) \quad \begin{cases} \mathbf{div}_x\, t = \dfrac{\partial t_{xx}}{\partial x} + \dfrac{\partial t_{xy}}{\partial y} + \dfrac{\partial t_{xz}}{\partial z}, \\[2mm] \mathbf{div}_y\, t = \dfrac{\partial t_{yx}}{\partial x} + \dfrac{\partial t_{yy}}{\partial y} + \dfrac{\partial t_{yz}}{\partial z}, \\[2mm] \mathbf{div}_z\, t = \dfrac{\partial t_{zx}}{\partial x} + \dfrac{\partial t_{zy}}{\partial y} + \dfrac{\partial t_{zz}}{\partial z}. \end{cases}$$

We may speak of the *flux of a tensor*, just as we do of the flux of a vector. By the tensor flux through a surface-element, of area df and with a normal unit-vector **n**, we mean the product of df and the vector which results from the multiplication of the components of the tensor by the unit-vector **n**, in accordance with the corresponding scheme of § 26 eqn. (12). The vector thus obtained may be called the *normal component of the tensor* relative to the surface-element, and may be denoted by the symbol $\mathbf{t}_n$. It must be remembered that $\mathbf{t}_n$ is not a scalar, but (as the Clarendon type indicates) a *vector*. We therefore have for its x-component

(5) $\quad (\mathbf{t}_n)_x = t_{xx}\,(\cos \mathbf{n},\, x) + t_{xy}\,(\cos \mathbf{n},\, y) + t_{xz}\,(\cos \mathbf{n},\, z),$

and two analogous expressions for the y- and the z-components.

Thus, by § 33 eqn. (3),

$$(6) \qquad \int (\mathbf{t}_n)_x\, df = \int \left(\frac{\partial t_{xx}}{\partial x} + \frac{\partial t_{xy}}{\partial y} + \frac{\partial t_{xz}}{\partial z} \right) d\tau.$$

Writing down the analogous equations for the y- and the z-components, multiplying the three equations by **i**, **j**, **k** respectively, and adding the results together, we obtain a relation analogous to Gauss's theorem, *viz.*,

$$(7) \qquad \int \mathbf{t}_n\, df = \int \mathbf{div}\, t\, d\tau.$$

The *integral of the normal component of a tensor, taken over a closed surface*, is equal to the *volume-integral of the vector divergence of the tensor*, taken over the total volume enclosed by the surface.

GENERAL THEORY OF VIBRATIONS AND OF WAVES

§ 37. Scalar Vibrations.

MANY of the physical processes which we observe in nature are peculiar in that a definite physical condition constantly *recurs* in them, after equal intervals of time. Such processes are called *periodic*, and the time within which the process runs its course is termed the *period*. This periodically varying state may be quantitatively determined by a scalar or a vector quantity which, in a truly periodic process, must be such a function of the time that it assumes the same value for every value of the time which constitutes a whole multiple of the period.

We will first consider a *scalar* quantity S, for the variations of vector quantities can always be referred to those of scalar quantities. If the variability of S with respect to the time be given by the equation

$$(1) \qquad S = \varphi(t),$$

then the relation

$$(2) \qquad \varphi(t_1 + n\tau) = \varphi(t_1)$$

must hold good, where τ is the period and n any integer. If the process always repeats itself in exactly the same way, and if the values of the function be S_1 at the time t_1, S_1' at the time $(t_1 + dt)$, S_1'' at the time $(t_1 + 2\,dt)$, and so on, then the value of S at the time $(t_1 + n\tau + dt)$ must again be S_1', that at the time $(t_1 + n\tau + 2\,dt)$ again S_1'', *etc.* In the case of a truly periodic process the following equations must therefore be satisfied :—

$$(3) \qquad \left\{ \begin{array}{ccc} \left|\dfrac{d\varphi}{dt}\right|_{t\,=\,t_1+n\tau} & = & \left|\dfrac{d\varphi}{dt}\right|_{t\,=\,t_1}, \\[2ex] \left|\dfrac{d^2\varphi}{dt^2}\right|_{t\,=\,t_1+n\tau} & = & \left|\dfrac{d^2\varphi}{dt^2}\right|_{t\,=\,t_1}, \quad etc. \end{array} \right.$$

132

Now we know that the *simplest* periodic functions are *sine* and *cosine functions*; for

$$\sin (x + 2n\pi) = \sin x ,$$
$$\cos (x + 2n\pi) = \cos x ,$$

and, furthermore, all the *derivatives* of both these functions are again simply sine or cosine functions. Hence both the functions $\sin x$ and $\cos x$ actually possess the properties required by eqns. (2) and (3), provided that we define x to be such a function of the time that an increase in x by 2π corresponds to an increase in t by the period τ. That is, we must put

$$(4) \qquad x = \frac{2\pi t}{\tau} ,$$

and we then obtain, as a mathematical expression of the *simplest* periodic process, the equation

$$(5) \qquad S - S_0 = A \sin \frac{2\pi t}{\tau} ;$$

(sine and cosine functions are, in this respect, completely equivalent, as we shall show immediately). In this equation S_0 denotes the arithmetical mean of the greatest and smallest values of the variable quantity; and the time is so measured that, at the commencement (*i.e.*, where $t = 0$), the variable S has the value S_0, while dS/dt is positive. If, in particular, $S_0 = 0$, *i.e.*, if the mean value of the variable quantity coincides with the zero point on the scale wherewith it is measured, eqn. (5) assumes the still simpler form

$$(6) \qquad S = A \sin \frac{2\pi t}{\tau} .$$

A process which can be described by means of this equation is termed a *harmonic vibration*, or a *pure sine vibration*, of the physical quantity S; or, simply, a vibration.[1] The difference between the value of the variable S at any moment and that of the quantity S_0, or the zero value, as the case may be, is called the *displacement*: [2] the latter may thus be positive or negative. The constant A, which represents the maximum value of the displacement on either side, is termed the *amplitude*: the period τ is also called the *time of vibration*,[3]

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[1] An alternative term for vibration is *oscillation*.

[2] Or " extension," as applied to springs.

[3] The *half* of this time is occasionally so designated, the time of vibration then being defined as that period elapsing between two successive moments when S is equal to S_0: the fact that the derivative dS/dt is then alternately positive and negative is left out of account. A *complete* vibration represents the interval after which the value, not only of S, but also of dS/dt, repeats itself with the same sign.

and its reciprocal value, which gives the number of vibrations in a unit of time, is known as the *vibration number* or *frequency*.

It follows from eqn. (6) that the oscillating quantity S may be represented, not only as a function of the time, but equally well as a function of an *angle* ψ which varies periodically with the time. That is to say, we have the relations

(7) $S = A \sin \psi$

or

$$S - S_0 = A \sin \psi,$$

as the case may be: the quantity ψ, which is sufficiently defined by any value between 0 and 2π, is termed the instantaneous *phase* of the vibration. Since it is a function of the time, its value depends upon that of the phase at the time $t = 0$. Where, as was assumed, the zero point of the time scale is so chosen that, at that moment, the phase of the vibration is likewise zero, the simple formula (7) holds good. If we are considering only one single vibration, we may naturally always choose the time scale so that, at the time $t = 0$, the phase vanishes. When, however, we have to take into account two or more simultaneous vibrations, this is, in general, no longer possible; and, if at the time $t = 0$ the phase of a vibration be equal to ϵ, it is at once clear that eqn. (7) will be replaced by the formula

$$S = A \sin (\psi + \epsilon)$$

or, in accordance with eqn. (6),

(8) $S = A \sin \left(\dfrac{2\pi t}{\tau} + \epsilon \right).$

The quantity ϵ, which determines the phase at the time $t = 0$, is known as the *phase-constant* of the vibration. It corresponds to an interval of time

(9) $t_0 = - \dfrac{\epsilon \tau}{2\pi},$

by which that point of time where S becomes zero appears displaced, relatively to the zero point on the time scale. Thus we may also write eqn. (8) in the form

(10) $S = A \sin \dfrac{2\pi (t - t_0)}{\tau}.$

The difference between the phase-constants of two vibrations, relative to the same system of time measurement, is called their *difference in phase*. Since two vibrations the

phase-difference of which amounts to a whole multiple of 2π are *identical*, a range of angles from $-\pi$ to $+\pi$ (or from 0 to 2π) suffices to determine any phase-difference.

Two vibrations whose phase-difference is equal to π are called *opposed*, for, although their displacements vanish simultaneously, at the moment when the displacement in the one vibration is greatest on the one side, it is a maximum on the opposite side in the other. If the amplitudes of the two vibrations are equal, the displacements of the two are always opposite and equal. When the phase-difference of two vibrations is $\pi/2$, the maximum displacement on one of either side will occur in the one whenever the displacement in the other vanishes.

Since

$$\cos \psi = \sin \left(\psi + \frac{\pi}{2} \right),$$

we see that a harmonic vibration may be equally well described by the formulæ

(11) $$S = A \cos \left(\frac{2\pi t}{\tau} + \epsilon \right)$$

or [4]

(12) $$S = A \cos \frac{2\pi (t - t_0)}{\tau},$$

as by eqns. (8) or (10). If we so measure the time that, at the time $t = 0$, the displacement is a maximum on the positive side, we have simply

(13) $$S = A \cos \frac{2\pi t}{\tau},$$

corresponding to eqn. (6).

§ 38. The Differential Equation of a Vibration.

Since it is a property of sine and cosine functions that they are equal to their second derivatives, taken with opposite sign, it follows that any harmonic vibration of a quantity S can be described by a *differential equation* of the form

(1) $$\frac{d^2S}{dt^2} = - a^2 S,$$

seeing that it can be represented as a sine or cosine function of the time. We make the proportional factor on the right-

[4] The value of ϵ in this equation naturally differs from that in eqn. (8) by $\pi/2$, when the same process is being described.

hand side of this equation equal to the *square* of any arbitrary real quantity, in order to express the fact that this right-hand side must always have the opposite sign to that of S itself. Inserting in eqn. (1) the value for S given by § 37 eqn. (8), we find for the constant a the value

$$(2) \qquad a = \frac{2\pi}{\tau}.$$

An equation of the form of eqn. (1) is termed a *differential equation of the second order*, because the highest derivative in which the unknown function of the time (*viz.*, S) appears is the second. It is called *linear*, because S and its derivatives occur in no higher degree than the first; and, finally, it is said to be *homogeneous*, for no term occurs in it that does not contain S. It is a special property of the above linear, homogeneous, differential equation of the second order that the coefficients of the unknown function and of its derivatives are not functions of the time, but are constants.

From what has been said it is clear that the differential equation (1) must possess a solution of the form

$$(3) \qquad S = A \sin (at + \epsilon),$$

which equation describes a harmonic vibration of period $2\pi/a$. It only remains to be seen whether this solution, in which the amplitude A and the phase-constant ϵ are arbitrary constants, is actually the *most general solution*; and consequently whether the quantity S, given as a function of the time in eqn. (3), represents the *complete integral* of the differential equation.

The complete integral of a differential equation must naturally contain as many undetermined *constants of integration* as render it possible to *adapt* the general solution *to the initial conditions*, which may be prescribed for the case that the independent variable (here the time) is *zero*. In the present case *two* integration constants are necessary for the determination of the original state, *viz.*, the values assumed at the time $t = 0$ by the quantity S and by its first derivative (dS/dt). Now, since eqn. (3) contains two undetermined integration constants, A and ϵ, it does actually represent the complete integral of the differential equation (1).

We can transform this complete integral by developing the expression $\sin (at + \epsilon)$ in accordance with the well-known trigonometrical formula for the sine of the sum of two angles. We then have

$$S = A \sin at \cos \epsilon + A \cos at \sin \epsilon,$$

or, by inserting for brevity the one constant product

(4) $$A \sin \epsilon = G$$

and the other constant product

(5) $$A \cos \epsilon = H,$$

we have

(6) $$S = G \cos at + H \sin at.$$

In this expression G and H are two undetermined constants whose values are first defined when the initial conditions are laid down ; for G and aH are actually the respective values assumed by the quantity S and its first derivative dS/dt at the time $t = 0$.

As both the functions $\sin at$ and $\cos at$ represent so-called *particular integrals*, it follows from eqn. (6) that we can obtain the general solution of eqn. (1) by multiplying each of these two particular integrals by an entirely *arbitrary* constant, and by then adding the two products together—a rule which, in a generalized form, holds good for all differential equations. In consequence of this, an expression of the form of eqn. (6) will also satisfy the differential equation (1), even when H is an *imaginary* quantity. Giving G the value 1, and H the value i, we thus see that

(7) $$J = \cos at + i \sin at$$

must also be a *particular integral* of the differential equation, as we may at once prove by inserting this value for S.

Now the particular integral (7) can be easily reduced to a much simpler form, which we shall find to be very convenient and advantageous for various calculations. We start off from the well-known *series* (deduced from *Maclaurin's Theorem*).

(8) $$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$

and

(9) $$\sin x = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots$$

Since

$$i^2 = -1, \quad i^3 = -i, \quad i^4 = +1, \text{ etc.,}$$

we have

(10) $$\cos x + i \sin x = 1 + \frac{ix}{1!} + \frac{i^2 x^2}{2!} + \frac{i^3 x^3}{3!} + \frac{i^4 x^4}{4!} + \cdots$$

Again

(11) $e^y = 1 + \dfrac{y}{1!} + \dfrac{y^2}{2!} + \dfrac{y^3}{3!} + \dfrac{y^4}{4!} + \ \cdot \ \cdot \ \cdot \ ;$

and, comparing this with eqn. (10), we obtain the following important relation which is called, after its discoverer, *de Moivre's Formula* :—

(12) $\cos x + i \sin x = e^{ix}.$

Hence we may write eqn. (7) in the form

(13) $J = e^{iat};$

and, if we give this value to S in eqn. (1), we see that J does actually form a particular integral of the differential equation. For we have

(14) $\dfrac{d^2 J}{dt^2} = - a^2 e^{iat} = - a^2 J.$

Eqn. (6) shows us that, in order to obtain the complete integral from the particular integral J, we have only to transform the latter by means of de Moivre's formula, *separate* the *real* and *imaginary parts* from each other, multiply each by an arbitrary constant (omitting the factor i), and then add the two together.

On the other hand, since a particular integral must also be represented by any function resulting from the multiplication of J by a constant factor, we obtain such an integral on multiplying J by the factor $e^{i\epsilon}$, where ϵ is any arbitrary constant which we can adapt to the prescribed initial conditions. The function

(15) $J' = e^{i(at + \epsilon)}$

therefore also represents a solution of the differential equation ; and we may arrive at the complete integral [in the form of § 37 eqns. (8) or (11)] by resolving the right-hand side of eqn. (15), by means of de Moivre's formula, into a real and an imaginary component, and then multiplying either the former or the latter by an arbitrary factor (which will represent the amplitude).[1]

§ 39. Vector Vibrations.

To the scalar differential equation discussed in the last section corresponds a *vector* differential equation of the form

(1) $\dfrac{d^2 \mathbf{V}}{dt^2} = - a^2 \mathbf{V}.$

[1] The values of the phase-constants will naturally differ by $\pi/2$ in the two cases.

It follows from the considerations of the previous section that its solution is given by

(2) $$\mathbf{V} = \mathbf{A} \sin at + \mathbf{B} \cos at ,$$

where $\mathbf{A}$ and $\mathbf{B}$ are vector constants. A process which is described by eqns. (1) or (2) is termed a *vector vibration*.

We see by eqn. (2) that the vibrating vector must always remain in the *plane* defined by the two constant vectors $\mathbf{A}$ and $\mathbf{B}$. If we therefore suppose the vibrating vector $\mathbf{V}$ to be drawn from any given origin, *its extremity will describe a plane curve*, the nature of which we can easily determine.

We will choose the direction of the vector $\mathbf{V}$ at the time $t = 0$ as the x-axis of a plane coordinate system, the plane of which coincides with that of the vibration of $\mathbf{V}$. By eqn. (2), the direction of x coincides with that of the vector $\mathbf{B}$, so that B_y vanishes. Further, we will put

(3) $$A_x = F, \quad B_x = G, \quad A_y = H .$$

Then eqn. (2) may be written in the analytical form

(4) $$\begin{cases} V_x = F \sin at + G \cos at, \\ V_y = H \sin at . \end{cases}$$

Hitherto we have taken the *zero point on the time scale* as being quite arbitrary, but we will now so choose it that, at the time $t = 0$, the *periodically varying magnitude of the vibrating vector* has an *extreme value*. We then have

(5) $$\left| \frac{dV}{dt} \right|_{t=0} = 0 .$$

But

$$V^2 = V_x{}^2 + V_y{}^2 ,$$

and hence

(6) $$V \frac{dV}{dt} = V_x \frac{dV_x}{dt} + V_y \frac{dV_y}{dt} .$$

In consequence of eqn. (5), the left-hand side of eqn. (6) vanishes for $t = 0$. The last term also vanishes for $t = 0$, seeing that, by eqn. (4), V_y is then equal to zero ; and since V_x therefore represents the magnitude of the vector at the time $t = 0$, it follows from eqn. (6) that dV_x/dt must also vanish for $t = 0$. But, by eqn. (4), $dV_x/dt = aF$ when $t = 0$, and the derivative dV_x/dt can therefore only vanish if the constant F is equal to zero. Hence, by choosing the arbitrary zero point on the time scale so that the magnitude of the vibrating vector has an extreme value, the constant F

becomes equal to zero, and eqn. (4) can be reduced to the simple form

(7) $$V_x = G \cos at, \quad V_y = H \sin at.$$

Now, since

$$\cos^2 at + \sin^2 at = 1,$$

we find from eqn. (7) that

(8) $$\frac{V_x^2}{G^2} + \frac{V_y^2}{H^2} = 1.$$

Since G and H are constants, we see from this equation that the plane curve described by the extremity of the vibrating vector must be an *ellipse*. Eqn. (1) therefore represents, in general, an *elliptical vector vibration*.

In the special case that either G or H is equal to zero, the ellipse becomes a straight line, and the vector vibration is then said to be *linear*: only in the case of a linear vibration can the magnitude of a vector periodically vanish. When, on the other hand,

(9) $$H = \pm G,$$

the ellipse becomes a circle, and we speak of a *circular* vibration. The vibrations of the components V_x and V_y then have the same amplitude, but differ in phase by $\pi/2$, the vibration of V_x *leading* or *lagging* with respect to that of V_y by a *quarter-period*, according as the positive or the negative sign in eqn. (9) is taken.[1] When G and H have the same sign, the direction of the vector **V**, in the course of its vibration, is first that of the positive x-axis, and a quarter-period later that of the positive y-axis. The circular vibration has then the same sense as a rotation which would carry the direction of the positive x-axis into that of the positive y-axis, by the shortest way. If, on the other hand, G and H are opposite in sign, the circular vibration has the opposite sense of rotation.

The special form of eqn. (7) holds for an elliptical vibration only when we so choose the arbitrary zero point of the time scale that the magnitude of the vibrating vector has an extreme value, *i.e.*, when the coordinate axes coincide with the axes of the ellipse. Otherwise the general form of eqn. (4) holds good. It may, however, easily be shown that, quite generally, an *elliptical vibration* can always be *resolved into two linear vibrations in any two directions which are*

---

[1] For $\cos x = \sin \left(x + \frac{\pi}{2} \right)$.

mutually at right angles. Just as in § 38, we only require to introduce two quantities D and ϵ, which are given by the relations [2]

(10) $F = D \cos \epsilon, \quad G = D \sin \epsilon,$

for eqn. (4) to assume the form

(11) $V_x = D \sin (at + \epsilon), \quad V_y = H \sin at.$

Thus the two linear, mutually perpendicular vibrations into which the elliptical vibration is resolved, as above, *differ* in general not only in amplitude, but also in *phase*.

Conversely, we can always *compound* two mutually perpendicular *linear vibrations* of the same period, but of different amplitude and with any difference in phase, into an *elliptical vibration*. In the particular case for which the amplitudes of the two vibrations are identical, and the phase-difference is exactly a *quarter-period* (or an odd multiple thereof), the resultant vibration becomes *circular*. Furthermore, when the phase-difference is a whole number multiple of a *half-period*, composition results in a *linear vibration*, even when the amplitudes are different.

Finally, on compounding two *opposite circular vibrations* of the same amplitude and period, we obtain a *linear* vibration whose amplitude is twice that of the circular vibrations, as follows from eqns. (7) and (9). Its direction is determined by the two diametrically opposed points on the circle where the opposite circular vibrations are constantly meeting. Hence, we may always compound two opposite and equal circular vibrations into a linear one, and, conversely, we may always resolve a linear vibration into two opposite and equal circular vibrations.

§ 40.   Damped Vibrations.

Although, in accordance with the simple theory of harmonic vibrations, the displacement attained at the end of every period should remain continually the same, yet, in all *actual* vibratory processes occurring in nature, we observe a phenomenon which is called the *decay of the vibrations*. The amplitude diminishes from one vibration to another until, after some time, the vibrations cease completely. The question therefore arises as to the way in which the *differential equation* of a harmonic vibration must be *modified*, in order to describe a decaying vibration.

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[2] Putting $D^2 = F^2 + G^2$, and $\tan \epsilon = \dfrac{G}{F}$.

To answer this question we will first take the especially simple example of a vibratory process in which the harmonically vibrating magnitude is the *distance* of a particle, moving in a straight line, from a fixed point in that line. If x be this distance, we have [by § 38 eqn. (1)], the differential equation

$$(1) \qquad \frac{d^2x}{dt^2} = -a^2x.$$

On the other hand, it follows from Newton's Second Law of Motion that the effective force (K) is given by

$$(2) \qquad K = m\frac{d^2x}{dt^2},$$

where m is the mass of the particle. We thus see that a particle moving in a straight line will execute a linear harmonic vibration [1] about a fixed point in this line, when it is attracted towards this point by a *force* which is *directly proportional to the distance* of the particle from the point.

A vibration such as is described by eqn. (1) will naturally continue unchanged, without diminution in amplitude. Actually, however, we observe in the course of vibrations of this sort (which, for example, are executed with fair approximation by a pendulum, as in § 6, provided that the displacement is small) that the *amplitude diminishes* from one vibration to the next, and that the originally vibrating body gradually comes *to rest*. We perceive that the cause of this lies in the *resistance of the air*.

Now we may take the resistance of the air into account by putting, in accordance with § 5 eqn. (13),

$$(3) \qquad m\frac{d\mathbf{v}}{dt} = \mathbf{K} - \kappa\mathbf{v},$$

where κ denotes the coefficient of frictional resistance. If, therefore, a particle is executing a linear vibration about a position of rest, but in a resisting medium which " *damps* " its vibrations, we find for its equation of motion

$$(4) \qquad \frac{d^2x}{dt^2} = -a^2x - k\frac{dx}{dt},$$

where k is written for κ/m.

---

[1] We speak simply of vibration instead of vibratory motion, because vibratory motions were first theoretically investigated, whence, by a generalization of the properties of such motion, the general notion of vibration was conceived.

Correspondingly, any process which can be represented by a differential equation of the form

(5) $$\frac{d^2S}{dt^2} + k\frac{dS}{dt} + a^2S = 0$$

is defined as a *damped vibration*, and the factor k is called the *constant cf damping*. Obviously the undamped, pure harmonic vibration, which has been treated in the previous sections, simply represents a special case of damped vibration, in which the constant of damping vanishes.

The question now arises as to what course is taken by a process such as is described by an equation of the form of eqn. (5), in which we again recognize a *homogeneous linear differential equation of the second order*. We see at once that we obtain a *particular integral* of this equation by putting

(6) $$S = e^{\lambda t},$$

where λ satisfies the *quadratic equation*

(7) $$\lambda^2 + k\lambda + a^2 = 0:$$

the *roots* of this equation are

(8) $$\lambda = -\frac{k}{2} \pm \sqrt{\frac{k^2}{4} - a^2}.$$

Denoting these two values of λ by λ_1 and λ_2, we find as the general solution, which will, of course, contain *two arbitrary constants*,

(9) $$S = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}.$$

Now when $k \geqq 2a$, that is to say, when the *damping* is sufficiently *strong*, λ_1 and λ_2 are *real*; and, since the absolute magnitude of the square root must always be smaller than $k/2$, λ_1 and λ_2 are necessarily negative. Hence in this case no trigonometric functions appear in the solution: the process then takes an *aperiodic* course (which we need not further investigate here), and the motion is not vibratory at all.

On the other hand, when $k < 2a$, the square root in eqn. (8) is *imaginary*, and λ is therefore *complex*. Writing for short

(10) $$a^2 - \frac{k^2}{4} = \beta^2,$$

we have

(11) $$\lambda = -\frac{k}{2} \pm i\beta.$$

We see by eqn. (9) that the value of S is given by

$$(12) \qquad S = e^{-\frac{k}{2}t} \left[C_1 e^{i\beta t} + C_2 e^{-i\beta t} \right].$$

If we now *split* the expression $e^{i\beta t}$ into a real and an imaginary part in accordance with de Moivre's formula, it follows from § 38 that each separate part (omitting the factor i) represents a particular integral.   Putting

$$C_1 + C_2 = G$$
and
$$C_1 - C_2 = H,$$

we thus obtain the general solution of eqn. (5) in the following form containing *two arbitrary constants* :—

$$(13) \qquad S = e^{-\frac{k}{2}t} (G \cos \beta t + H \sin \beta t).$$

Putting again

$$G = A \sin \epsilon$$
and
$$H = A \cos \epsilon,$$

we have, in accordance with the formula for the sine of the sum of two angles,

$$(14) \qquad S = A e^{-\frac{k}{2}t} \sin (\beta t + \epsilon).$$

The phase-constant ϵ vanishes if the time be measured from the moment when S is equal to zero : in this special case eqn. (14) assumes the simpler form

$$(15) \qquad S = A e^{-\frac{k}{2}t} \sin \beta t.$$

We may thus obtain the value of S at any moment during the damped vibration by diminishing, in the proportion $1 : e^{\frac{k}{2}t}$, the corresponding displacement which would occur at the same moment during an imaginary undamped vibration, of period $2\pi/\beta$ and amplitude A. (*Cf.* Fig. 31,
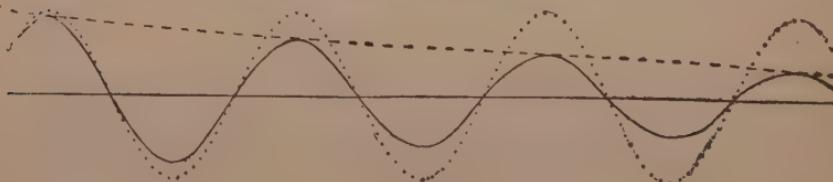

Fig. 31.

where the ordinate S is represented as a function of the abscissa t. The dotted curve denotes the undamped vibration of period $2\pi/\beta$, and the full curve the damped vibration.)

In a damped vibration the period remains constant, just as in an undamped vibration; *i.e.*, the course of the damped vibration is also what we term *isochronous*. The amplitude certainly diminishes, but the velocity with which the ever-decreasing vibrations take place also diminishes. The period is

$$(16) \qquad \tau = \frac{2\pi}{\beta} = \frac{2\pi}{\sqrt{a^2 - \dfrac{k^2}{4}}};$$

and it therefore appears lengthened relatively to the period $2\pi/a$, proper to an undamped vibration with the same value of a. Thus the stronger the damping, the *slower* the *vibration* becomes.

The amplitude diminishes from one vibration to the next in the proportion $1 : e^{\frac{k}{2}\tau}$. The quantity

$$(17) \qquad \frac{k}{2}\tau = \sigma$$

is known as the *logarithmic decrement of the vibration*, because the difference between the logarithms of two successive maximum displacements is equal to σ. Conversely, we can deduce the constant of damping (k) by observing the difference of two successive maximum displacements.[2]

§ 41. Forced Vibrations.

The phenomena which occur when a certain vibratory process is *influenced by another, likewise periodic*, process are of great importance for many branches of physics. An especially simple instance of this kind is afforded by a particle which is executing a linear vibration about a position of rest, and upon which acts a force that is itself a periodic function of the time. Where, in particular, the variation of the force relative to the time may be represented by a sine function, *e.g.*, by the formula

$$(1) \qquad K = K_1 \sin pt\,,$$

we find for the equation of motion of the particle, denoting the quotient K_1/m by P,

$$(2) \qquad \frac{d^2x}{dt^2} + k\frac{dx}{dt} + a^2x = P \sin pt\,.$$

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[2] It should be borne in mind (though we shall not go into the point here) that the instant of maximum displacement does not correspond exactly to the phase $\pi/2$ (when $\epsilon = 0$), but to a somewhat smaller phase; the stronger the damping, the greater is this shift.

We are here taking the general case of a damped vibration : the results obtained will naturally hold for an undamped vibration, if k be put equal to zero.

In general, therefore, any sinusoidal vibratory process which is subjected to some external influence, whose value is itself a harmonic function of the time, may be described by an equation of the form

$$(3) \qquad \frac{d^2S}{dt^2} + k\frac{dS}{dt} + a^2 S = P \sin pt \,,$$

the arbitrary zero of time being so chosen that $P \sin pt$ then vanishes, while its time-derivative is positive. In order to investigate the properties of such a vibration, subjected to some periodic external influence, it is only necessary to find the complete integral of eqn. (3). This equation resembles the differential equation of a free harmonic vibration in that it is a *linear differential equation of the second order* with constant coefficients, but it differs in being *non-homogeneous*, for the term on the right-hand side does *not contain* the unknown function S.

Now the integration of eqn. (3) may be *reduced* to that of a *homogeneous* differential equation in which the left-hand side agrees with the left-hand side of eqn. (3), but whose right-hand side is zero, *i.e.*, which has the form

$$(4) \qquad \frac{d^2S}{dt^2} + k\frac{dS}{dt} + a^2 S = 0 \,.$$

For, if S_1 and S_2 are any two *particular integrals* of the non-homogeneous eqn. (3), the following equations must be satisfied for any value whatsoever of the time :—

$$(5) \qquad \begin{cases} \dfrac{d^2S_1}{dt^2} + k\dfrac{dS_1}{dt} + a^2 S_1 = P \sin pt \,, \\[2mm] \dfrac{d^2S_2}{dt^2} + k\dfrac{dS_2}{dt} + a^2 S_2 = P \sin pt \,. \end{cases}$$

Consequently, the equation obtained by *subtracting* the second of the above two equations from the first must also be satisfied for any arbitrary time, *i.e.*,

$$(6) \quad \frac{d^2}{dt^2}(S_1 - S_2) + k\frac{d}{dt}(S_1 - S_2) + a^2 (S_1 - S_2) = 0.$$

Hence the expression $(S_1 - S_2)$ represents an integral of the homogeneous differential equation (4) ; conversely, we have the following rule which is important for later considerations :—

When a particular integral of the non-homogeneous equation is known, as well as a particular integral of the homogeneous equation, the sum of the two also represents an integral of the non-homogeneous differential equation.

On the other hand, we require *only two* arbitrary constants of integration in order to be able to adapt the general solution of the homogeneous eqn. (4) to the prescribed initial conditions. Now the general solution of eqn. (4) is

$$(7) \qquad S' = e^{-\frac{k}{2}t} A \sin (\beta t + \epsilon)$$

[§ 40 eqn. (14)], where

$$\beta = \sqrt{a^2 - \frac{k^2}{4}}.$$

Since this solution already contains *two* arbitrary integration constants, *viz.*, A and ϵ, we have only to find *one single particular integral of the non-homogeneous differential equation*; moreover, if this be S'', it follows from what was said above that the sum

$$(8) \qquad S = S' + S''$$

represents the *complete general solution* of the non-homogeneous eqn. (3). Naturally, the constants occurring in the particular integral S'' can no longer be regarded as arbitrary, but must be expressed in terms of k, a and p in such a way that the sum $(S' + S'')$ satisfies eqn. (3).

In attempting to find the particular integral S'' let us begin by taking, *as a trial solution*, a harmonically vibrating quantity with the same period as the external influence; that is to say, we put

$$(9) \qquad S'' = R \sin (pt + \delta),$$

where the amplitude (R) and the phase-constant (δ) are constants that we have to determine. Differentiating with respect to time, we have

$$(10) \qquad \begin{cases} \dfrac{dS''}{dt} = pR \cos (pt + \delta), \\[2mm] \dfrac{d^2S''}{dt^2} = -p^2R \sin (pt + \delta). \end{cases}$$

Furthermore, we may transform the right-hand side of eqn. (3) in accordance with the identity

$$(11) \begin{cases} \sin pt = \sin \left[(pt + \delta) - \delta \right] = \sin (pt + \delta) \cos \delta \\ \qquad\qquad - \cos (pt + \delta) \sin \delta. \end{cases}$$

Inserting the values given by eqns. (10) and (11) in eqn. (3), we find

$$(12) \quad \left\{ \begin{array}{l} \sin{(pt + \delta)} \left[-p^2 R + a^2 R - P \cos{\delta} \right] \\ \quad + \cos{(pt + \delta)} \left[kpR + P \sin{\delta} \right] = 0 \,. \end{array} \right.$$

If, now, the function S'', which we have taken as a *trial solution* of the non-homogeneous differential equation, is to represent an *actual* integral of the latter, eqn. (12) must be satisfied for *any* value whatever of the independent variable t, seeing that it has been obtained by inserting the value of S'' in the differential equation. It must be satisfied both for a value t_1, where

$$(13) \qquad\qquad pt_1 + \delta = 0 \,,$$

and for a value t_2, where

$$(14) \qquad\qquad pt_2 + \delta = \frac{\pi}{2} \,.$$

Putting t equal to t_1 in eqn. (12), we see that the expression in the second square brackets must vanish. Again, putting t equal to t_2, the expression in the first square brackets must vanish. Hence eqn. (12) can only be satisfied for *all* values of the time, if each of the expressions in square brackets in eqn. (12) vanishes by itself. We must therefore have the relations

$$(15) \quad \left\{ \begin{array}{l} R \, (a^2 - p^2) = P \cos{\delta} \,, \\ \quad kpR = -P \sin{\delta} \,. \end{array} \right.$$

By means of these two equations we can always determine the constants R and δ, which we have previously left undetermined. Hence the function S'', which we took as a trial solution, actually does represent a particular integral of the non-homogeneous differential equation, when δ and R are given the values resulting from eqn. (15). We find in the first place

$$(16) \qquad\qquad \delta = \tan^{-1} \frac{kp}{p^2 - a^2} \,;$$

and, since

$$(17) \qquad \sin{\delta} = \frac{\tan{\delta}}{\sqrt{1 + \tan^2{\delta}}} = \frac{kp}{\sqrt{(p^2 - a^2)^2 + k^2 p^2}}$$

we obtain as the value of the constant R

$$(18) \qquad\qquad R = -\frac{P}{\sqrt{(p^2 - a^2)^2 + k^2 p^2}} \,.$$

It follows from eqns. (7), (8) and (9) that the most general solution of the non-homogeneous differential equation is given by

(19) $S = e^{\frac{k}{2}t} A \sin(\beta t + \epsilon) + R \sin(pt + \delta)$.

The process described by the non-homogeneous differential equation is therefore *compounded of two vibrations*. One of these is a *damped* vibration whose constants A and ϵ are given by the initial values for the time $t = 0$. It proceeds as if there were no external influence at all present, and is consequently termed the *natural vibration*. The second vibration is superimposed upon the gradually diminishing natural vibration, and has *the same period as the external influence*, or, as we may also express it, has the same period as the *exciting vibration*: it is therefore termed a *forced vibration*. Despite the damping present, it is *undamped* (as long as the external influence continues to vibrate harmonically), so that, after a sufficient length of time, the natural vibration originally present becomes unnoticeable, and only the forced vibration is thenceforth capable of observation.

The meaning of the formulæ which hold for the forced vibration becomes clearer if we replace the constants a and p by the natural frequency (for the case of infinitesimal damping), and the frequency of the exciting vibration. Denoting the former by ν_0 we have, by § 38 eqn. (2) (since the frequency is the reciprocal of the period),

(20) $a = 2\pi\nu_0$

and

(21) $p = 2\pi\nu$;

whence

(22) $\delta = \tan^{-1} \dfrac{k\nu}{2\pi(\nu^2 - \nu_0^2)}$

and

(23) $R = -\dfrac{P}{2\pi\sqrt{4\pi^2(\nu^2 - \nu_0^2)^2 + k^2\nu^2}}$.

Eqn. (23) shows us that the amplitude of the forced vibration is directly proportional to that of the exciting vibration, and that, for given values of k and P, and with weak damping, it attains its greatest value when ν is equal to ν_0, *i.e.*, when the period of the exciting vibration coincides with that of the natural one. (When the damping is strong,

this maximum does not occur when ν is equal to ν_0, but for a somewhat smaller value of the frequency.) The large increase in the amplitude for the case in which ν is equal to ν_0, or nearly so, is known as *resonance*. The smaller the damping, the greater becomes the amplitude of the resonant vibration, and, theoretically, it actually becomes infinite when k is equal to zero. The *displacement in phase* of the forced relative to the exciting vibration becomes all the greater, as eqn. (22) shows us, the smaller the difference between ν and ν_0. In the case of complete resonance it is equal to $\pi/2$, corresponding to a *quarter-period*; otherwise the displacement is greater the stronger the damping.

§ 42. Plane Waves.

There is, in many branches of physics, a class of processes in which there exists a particular *connection* between the *way in which the conditions at one and the same point O vary with respect to the time*, and the *continuous space-distribution of different conditions at one and the same moment*, along a straight line drawn from the point. In describing this connection we will first consider a simple *special case*.

On comparing with one another the conditions which are *simultaneously* present at any one moment (t_0) *along a straight line* drawn from the point O, we will suppose that at a certain distance, v units of length from the latter, we find a condition identical with that which existed at the point O one unit of time *previous* to the time t_0. At a distance equal to $2v$ units of length, we find a condition identical with that which existed at O two units of time before the time t_0, *etc.* Therefore, in general, the condition existing at the time t_0, at a distance x from O, is the same as that which had existed at O at the time $\left(t_0 - \dfrac{x}{v}\right)$. We accordingly find the condition existing at the point O at a given time to be repeated a unit of time later at a point distant v units of length from O. In other words, we may say that the *condition* is being *propagated* or *transmitted* with a velocity v along the straight line from the point O.

Such a propagation of a condition is called an *advancing plane wave* (the reason for the term "plane" will become evident later), and the quantity v is termed the *velocity of propagation of the wave*. The direction of the straight line, to which the connection under discussion is referred, is known as the *direction of propagation* of the wave.

We will now assume that the variable quantity of condition can be represented by a *scalar S* : the more general case, in which the condition is defined by a vector, can then be readily reduced to this simpler one.   Let the variation of the condition at the point O with respect to the time be described by an equation of the form

(1) $$S = \varphi(t),$$

where $\varphi(t)$ is an entirely *arbitrary* function of the time, which is, however, continuous and differentiable. Since at a point P distant x from O, every condition occurs x/v units of time *later* than at O, the course of the variations of condition at any *arbitrary* point P is given by the equation

(2) $$S = \varphi\left(t - \frac{x}{v}\right).$$

Eqn. (2) represents the general expression for a plane wave which is travelling *away from* the point O. If, on the other hand, the wave is *approaching* O, then the condition existing at O at the time t_0 has already occurred at the point P, distant x from O, x/v units of time *earlier*. Thus the condition at P at the time t_0 corresponds to that which will occur at O at the time $\left(t_0 + \frac{x}{v}\right)$. The variation of condition at the point P, when the wave is travelling *towards* O, is hence described by the equation

(3) $$S = \varphi\left(t + \frac{x}{v}\right).$$

Hitherto we have considered only the special case that the condition at the point P, at the time $\left(t_0 \mp \frac{x}{v}\right)$ is *identical* with that existing at the point O at the time t_0. In general, however, the *ratio* between these two quantities of condition is *not* necessarily *unity*, but, although quite independent of the time, it may be a definite *function of x*. Thus the *general* expression for a plane wave is given by an equation of the form

(4) $$S = \psi(x) \cdot \varphi\left(t \mp \frac{x}{v}\right).$$

When, in particular,

(5) $$\psi(x) = e^{-ax}$$

where a is a constant, we say that an *absorption* of the plane wave takes place. If $\psi(x)$ is equal to *unity*, we then have to

deal with the previously discussed case of an absorption-free plane wave.

An extremely important *special case* arises when the function $\varphi(t)$ is *periodic*, and when, in particular, $\varphi(t)$ represents a *harmonic vibration*, instead of being left quite arbitrary, as hitherto. We then have, according to § 37 eqn. (8),

$$(6) \qquad \varphi(t) = A \sin\left(\frac{2\pi t}{\tau} + \epsilon\right).$$

When $\varphi(t)$ has this form, the wave is then known as a *harmonic wave*, or as a *sine wave*, and it is described (provided it be plane as well) by the equation

$$(7) \qquad S = A \sin\left[\frac{2\pi}{\tau}\left(t \mp \frac{x}{v}\right) + \epsilon\right].$$

In this expression the negative or the positive sign is to be taken according as the wave is travelling away from, or towards, the point from which the distance x is measured.

As the same condition—and in the same sense of variation [1]—*recurs* at the point O after *periods* of duration τ, it follows that, at one and the same moment, the same condition must be *repeated* along the direction of propagation of the wave *at regular intervals*, each of length equal to the distance through which the wave travels in the time τ. Calling this length λ, we see that it must be connected with the quantity τ by the relation

$$(8) \qquad \lambda = v\tau :$$

alternatively, introducing the frequency ν which is the reciprocal of τ,

$$(9) \qquad v = \lambda\nu.$$

The quantity λ is defined as the wave-length. We may accordingly also write eqn. (7) in the form

$$(10) \qquad S = A \sin\left[2\pi\left(\frac{t}{\tau} \mp \frac{x}{\lambda}\right) + \epsilon\right].$$

If only a single wave be considered, we can always get rid of the phase-constant ϵ by suitably choosing the zero on the time scale.

On the other hand, when *two waves* of the same wave-length coming from different points O_1 and O_2 meet at a point P, where O_1P is equal to x_1 and O_2P to x_2, their *phases* will in general *differ* at P, even when their phase-

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[1] *I.e.*, not only must S have the same value, but also $\partial S/\partial t$ must have the same sign.

constants are the same. The difference in phase will then be given by

$$(11) \qquad \delta = 2\pi\left(\frac{x_2 - x_1}{\lambda}\right).$$

The quantity δ will consequently be equal to a whole multiple of 2π, so that the difference in phase vanishes, when the difference $(x_2 - x_1)$ amounts to a whole multiple of a wave-length: $(x_2 - x_1)$ is called the *difference in path* of the two waves. When the difference in path is equal to an odd multiple of half a wave-length, the difference in phase is equal to π, and the vibrations of the two waves at the point P are then opposed to each other.

Waves may naturally be *compounded* and *resolved* in the same way as vibrations. Special interest attaches to the phenomenon arising from the *superposition* of two waves of the same amplitude, period and phase, one of which, however, is travelling towards a point, whereas the other is travelling away from it. The resultant wave is then described by the equation

$$(12) \quad S = A \sin\left[2\pi\left(\frac{t}{\tau} - \frac{x}{\lambda}\right)\right] + A \sin\left[2\pi\left(\frac{t}{\tau} + \frac{x}{\lambda}\right)\right].$$

In accordance with the well-known trigonometrical formula

$$(13) \qquad \sin(\alpha - \beta) + \sin(\alpha + \beta) = 2 \sin\alpha \cos\beta,$$

eqn. (12) may also be written in the form

$$(14) \qquad S = 2A \cos\frac{2\pi x}{\lambda} \sin\frac{2\pi t}{\tau}.$$

The superposition of the two waves therefore gives rise to a state of vibration in which, at every moment, the vibration has *at all places the same phase*: for $t = 0, \tau/2, \tau, 3\tau/2$, *etc.*, S vanishes *everywhere*. What *differs* from place to place at the same time is not, as in the case of a travelling wave, the phase, but the *amplitude* of the vibration, which exhibits a *periodic* variation along the straight line in which the two individual waves are propagated. For, *at intervals of half a wave-length*, there are places for which, independently of the time,

$$\cos\frac{2\pi x}{\lambda} = 0.$$

At these places, in which x is accordingly an odd multiple of a quarter wave-length $(x = \lambda/4, 3\lambda/4, 5\lambda/4, etc.)$, S is thus *always zero*; no vibration at all takes place at these points, and they are consequently called *nodes*. In between these

places there are others for which x is equal to a multiple of half a wave-length ($x = 0$, $\lambda/2$, λ, $3\lambda/2$, 2λ, *etc.*), and for which, accordingly,

$$\cos \frac{2\pi x}{\lambda} = \pm 1 .$$

At these latter places the displacement at every moment is a *maximum*, both as regards its absolute amount and in its dependency on position. Such places are called *antinodes*. The greatest displacement which occurs at the antinodes, alternately on either side, and at intervals of time $\tau/2$, is, according to eqn. (14), *twice as great* as the amplitude of either individual wave.

The particular kind of waves described by eqn. (14) are termed *stationary waves*. With such waves, at a certain moment ($t = 0$), S is everywhere equal to zero. Subsequently S becomes positive in the interval between $x = 0$ and $x = \lambda/4$, but negative between $x = \lambda/4$ and $x = 3\lambda/4$, and so on, whereas at the actual points $x = \lambda/4$ and $x = 3\lambda/4$, S is c o n t i n u o u s l y zero (Fig. 32). At the time $t = \tau/4$
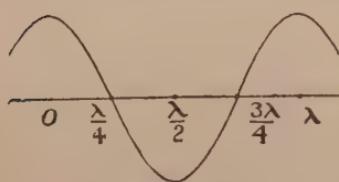

FIG. 32.

the displacement everywhere reaches a maximum ; then the absolute magnitude of S decreases until, at the time $t = \tau/2$, its value is again everywhere zero. After this S becomes negative between $x = 0$ and $x = \lambda/4$, as well as between $x = 3\lambda/4$ and $x = 5\lambda/4$, while between $x = \lambda/4$ and $x = 3\lambda/4$ it becomes positive, until, at the time $t = 3\tau/4$, the displacement is again a maximum throughout, but everywhere on the opposite side to that maintaining at the time $t = \tau/4$. Then the magnitude of the displacement again decreases until, at the time $t = \tau$, S is once more everywhere zero.

As regards the *differential equation of a plane wave*, we see by eqn. (10) that, just as in the case of the differential equation of a harmonic vibration [§ 38 eqn. (1)], the second derivative of S is always proportional to S itself with the sign reversed, provided that we consider only one given position and regard x as constant. We thus have

(15) $$\frac{\partial^2 S}{\partial t^2} = -\frac{4\pi^2}{\tau^2} S .$$

But we also have

(16) $$\frac{\partial^2 S}{\partial x^2} = -\frac{4\pi^2}{\lambda^2} S ;$$

so that, combining eqns. (15) and (16), and denoting the quotient λ/τ (which, by eqn. (8), is equal to v) by κ, we obtain the important relation

(17)
$$\frac{\partial^2 S}{\partial t^2} = \kappa^2 \frac{\partial^2 S}{\partial x^2}.$$

Conversely, it follows from previous considerations that every partial differential equation of the form of eqn. (17) possesses a *particular solution* of the form

(18)
$$S = A \sin\left[\frac{2\pi}{\tau}\left(t \mp \frac{x}{\kappa}\right) + \epsilon\right].$$

It is easy to see, moreover, that the more general functions represented by eqns. (2) and (3) also satisfy eqn. (17). In point of fact, the *most general solution* of the partial differential equation (17) is given by the expression

(19)
$$S = C_1 \cdot \varphi_1\left(t - \frac{x}{\kappa}\right) + C_2 \cdot \varphi_2\left(t + \frac{x}{\kappa}\right),$$

where φ_1 and φ_2 are two quite arbitrary functions, and C_1 and C_2 two arbitrary constants.[2]

§ 43. Spherical Waves.

Closely related to § 42 eqn. (17) is another *partial differential equation* which plays an important *rôle* in various branches of theoretical physics : this equation is

(1)
$$\frac{\partial^2 S}{\partial t^2} = \kappa^2 \nabla^2 S.$$

In order to obtain a *particular solution* for this differential equation we will begin by considering any point O, situated in the region wherein the equation is satisfied, and we will choose this point as the origin of a rectangular coordinate system. Let the distance of any other point from O be r. Then, by § 32 eqn. (20),

(2)
$$\nabla^2 S = \frac{\partial^2 S}{\partial x^2} + \frac{\partial^2 S}{\partial y^2} + \frac{\partial^2 S}{\partial z^2}.$$

Moreover,

(3)
$$r^2 = x^2 + y^2 + z^2,$$

---

[2] If we put $\phi_1(t) = \phi_2(t) = \sin\dfrac{2\pi t}{\tau}$, and $C_1 = C_2$, we obtain a harmonic *stationary* wave, which of course also represents a particular solution of the general differential equation. In this connection we may, perhaps, mention an important theorem of Fourier, according to which *any given form of wave* may be represented by the *superposition of a series of sine-waves*.

whence

(4)
$$\frac{\partial r}{\partial x} = \frac{x}{r}, \text{ etc.}$$

Now

(5)
$$\frac{\partial S}{\partial x} = \frac{\partial S}{\partial r} \cdot \frac{\partial r}{\partial x} = \frac{\partial S}{\partial r} \cdot \frac{x}{r};$$

and hence

(6)
$$\frac{\partial^2 S}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial S}{\partial r}\right) \cdot \frac{x}{r} + \frac{\partial S}{\partial r} \cdot \frac{\partial}{\partial x}\left(\frac{x}{r}\right).$$

Since, however, eqn. (5) is satisfied for any arbitrary scalar, and therefore for $\partial S/\partial r$, we have

(7)
$$\frac{\partial}{\partial x}\left(\frac{\partial S}{\partial r}\right) = \frac{\partial^2 S}{\partial r^2} \cdot \frac{x}{r}.$$

Again

(8)
$$\frac{\partial}{\partial x}\left(\frac{x}{r}\right) = \frac{1}{r} - \frac{x}{r^2} \cdot \frac{\partial r}{\partial x}.$$

Hence, substituting in eqn. (6) the values given by eqns. (7), (8) and (4), we find

(9)
$$\frac{\partial^2 S}{\partial x^2} = \frac{\partial^2 S}{\partial r^2} \cdot \frac{x^2}{r^2} + \frac{\partial S}{\partial r}\left(\frac{1}{r} - \frac{x^2}{r^3}\right).$$

Forming the analogous equations for $\partial^2 S/\partial y^2$ and $\partial^2 S/\partial z^2$ and adding all three equations together, we thus obtain, when we also take eqn. (3) into account,

(10)
$$\nabla^2 S = \frac{\partial^2 S}{\partial r^2} + \frac{2}{r}\frac{\partial S}{\partial r}.$$

Now

(11)
$$\frac{\partial(rS)}{\partial r} = S + r\frac{\partial S}{\partial r},$$

and consequently

(12)
$$\frac{\partial^2(rS)}{\partial r^2} = 2\frac{\partial S}{\partial r} + r\frac{\partial^2 S}{\partial r^2}.$$

We may thus write eqn. (10) also in the form

(13)
$$\nabla^2 S = \frac{1}{r} \cdot \frac{\partial^2(rS)}{\partial r^2};$$

and, since r is independent of t, eqn. (1) may be transformed into

(14)
$$\frac{\partial^2(rS)}{\partial t^2} = \kappa^2 \frac{\partial^2(rS)}{\partial r^2}.$$

If, now, we put (rS) equal to U, we recognize in eqn. (14) the partial differential equation which we have already treated in the previous section [§ 42 eqn. (17)], except that S is here replaced by the quantity U, and x by the quantity r. That is to say, we may now regard *any arbitrary* straight line, drawn in any direction from the point O, as the direction of propagation of a wave. We know, however, from § 42 eqn. (18) that eqn. (14) possesses a *particular solution* in the form

$$(15) \qquad U = A \sin \left[\frac{2\pi}{\tau} \left(t - \frac{r}{\kappa} \right) \right]$$

or

$$(16) \qquad S = \frac{A}{r} \sin \left[\frac{2\pi}{\tau} \left(t - \frac{r}{\kappa} \right) \right],$$

leaving the phase-constant out of account.

If this particular solution hold good, a *wave* may *travel uniformly in all directions* with a velocity κ *from any arbitrarily chosen point*, and in such a way that the scalar S everywhere executes a harmonic vibration. The *amplitude* of vibration, however, will *depend on position*, being *inversely proportional to the distance from the origin*. At every point on a *spherical surface* described about the point O as centre, the value of S at any given moment will be *the same*. A process of this kind, such as is represented by eqn. (16), is termed a *spherical wave*.

If we take into consideration only a part of the region embraced by a spherical wave, the linear dimensions of this part being *small* in comparison with its distance from the origin of the wave, we may regard the corresponding portions of the spherical surfaces in this region as *plane*. We can then construct a coordinate system, with any given point as origin, in such a way that the direction of the x-axis is that of the radius of the sphere, so that the y-z-plane thus coincides with a portion of a spherical surface which is to be regarded as plane. Since the value of S at any moment is the same everywhere on the spherical surface, we have, relatively to such a coordinate system,

$$(17) \qquad \frac{\partial S}{\partial y} = \frac{\partial S}{\partial z} = 0,$$

and therefore

$$\nabla^2 S = \frac{\partial^2 S}{\partial x^2}.$$

Hence the differential equation (1) assumes in this special case the simpler form

$$(18) \qquad \frac{\partial^2 S}{\partial t^2} = \kappa^2 \frac{\partial^2 S}{\partial x^2}.$$

This equation, however, is nothing else than the differential equation of a *plane wave*. Thus plane waves, the reason for whose name is evident from what has just been said, represent a *special case of spherical waves*.

What holds good for a scalar naturally holds in corresponding fashion also for a vector. Let us suppose that we have a *vectorial differential equation* of the form

$$(19) \qquad \nabla^2 \mathbf{A} = \frac{1}{\kappa^2} \frac{\partial^2 \mathbf{A}}{\partial t^2}.$$

Since we can resolve this equation into three scalar equations (for A_x, A_y, A_z), the existence of eqn. (19) at once shows us that an *undulatory propagation* of the vector $\mathbf{A}$ is always *possible*. The vector will then, in general, carry out an *elliptical vibration* which, in special cases, may become circular or linear.

Should these waves be *plane*, we find that they are distinguished by a particularly simple property in the special case that

$$(20) \qquad \operatorname{div} \mathbf{A} = 0.$$

If we choose the direction of propagation as the x-axis, it follows from eqn. (17) that $\partial A_y / \partial y$ and $\partial A_z / \partial z$ vanish, and the partial derivative $\partial A_x / \partial x$ must then also vanish, in accordance with eqn. (20). We thus have

$$(21) \qquad \nabla^2 A_x = 0.$$

But, when this is the case, we see by eqn. (19) that

$$(22) \qquad \frac{\partial^2 A_x}{\partial t^2} = 0.$$

Consequently, the x-component of the vector takes no part at all in the vibratory process. The *vibrations* take place *at right angles* to the direction of propagation of the wave, defined by the x-axis. The *plane waves of a vector whose divergence vanishes* are therefore always *purely transversal*.

Chapter VI

THE MOTION OF DEFORMABLE BODIES

§ 44.   The Conception of a Deformable Body.

THE conception of a system of discrete particles whose mutual distances are invariable proves to be sufficient and suitable only for a certain group of *actual* bodies, in the description of their *mechanical behaviour* ; for it presupposes that the body under consideration retains its *form unaltered*, even when acted upon by external forces.   Such is certainly the case, in very fair approximation, with many *solid* bodies ; but, on the other hand, the phenomena of *elasticity*, shown also by solid bodies, stand in contradiction to the simple picture of the rigid body.   This conception, moreover, becomes entirely insufficient when we are dealing with such bodies as have *no definite form* whatever, known as *fluids*.

We can, however, keep to our picture of a *system of discrete particles*, even in the case of a *deformable body*, provided that we relinquish the special postulate of the invariability of the internal distances, and replace it by more general *postulates*.   The latter must express the fact that the individual particles are not isolated from one another, but that they represent, taken as a whole, a *coherent body*.

We will consequently begin by assuming that the particles are so *closely* packed that, if P be any arbitrary geometrical point within the space occupied by the deformable body, the distances separating P from the particles in its immediate neighbourhood are always *very small* in comparison with the intervals which, with the required *degree of accuracy*, may still be regarded as *differentials of the coordinates* (dx, dy, dz) in our theoretical investigations.   A system of particles which satisfies this condition is known as a *continuously distributed* mass.

If, now, we cut off around the point P a volume whose linear dimensions are of the same order of magnitude as the differentials of the coordinates, this volume must contain such a large number of particles that, when the total mass of all the particles included in the volume is divided by the volume itself, the quotient may be regarded as *independent*

of the size of the volume. This quotient is called the *density* of the deformable body at the place in question ; but it must be borne in mind that the density at one and the same place may vary in value with the time.

To our *first postulate of the continuous distribution of mass* we will now add a second—that of *continuous motion.* By this we mean that the mass, which we have supposed to be continuously distributed, must move in such a way that interior portions of the body are thereby neither torn apart nor thrust into one another. These latter occurrences are only excluded if the *velocity* possessed by the various portions of the supposed continuously distributed mass, at any given moment, represents a *continuous, differentiable space-function.* Since the velocity of every single particle of the mass is further to be regarded as a continuous, differentiable *time-function* (just as with every individual material particle), the velocity within a deformable body appears as a continuous differentiable function of altogether *four independent variables viz.,* of x, y, z, and t.

The *forces* acting on a *delimited volume within the deformable body* may be divided into *two groups.* The first group is formed by those forces, such as, for example, the force of gravity, which represent *external forces* not merely for the delimited volume but for the *whole body.* The second group of forces that acts on the delimited volume from without consists of those forces which, although external as far as that volume is concerned, nevertheless represent *internal* forces for the body as a whole. That is to say, if we denote by I the particles which lie within the delimited volume, and by II those which lie outside the latter but within the deformable body, the second group is formed by the forces exerted by particles II upon particles I.

As regards these forces, the conception of the deformable body requires a *third postulate* to complete it, *viz.,* that they shall be *central forces,* but *effective only within distances of the same order of magnitude as the distance between two neighbouring particles.* At greater distances these central forces will have no effect ; and consequently, in accordance with what has been stated previously, they must be *inoperative* at distances of the same order of magnitude as the intervals which we are justified in regarding as differentials of the coordinates.

Now if this be the case, the total effect produced by particles II on particles I *reduces* to the one produced by those particles II lying *directly at the bounding surface* upon those

particles I situated likewise just at this boundary, and which are *contiguous* to the particles II and in their *immediate neighbourhood*.

Let us, therefore, cut out of the bounding surface a surface-element df, the linear dimensions of which shall be large as compared with the distance between two successive particles. If $d\mathbf{Q}$ be the total force exerted upon the particles I lying on this surface-element by the immediately adjacent particles II, it follows again from the postulate of the continuous distribution of mass that the quotient $d\mathbf{Q}/df$ may be regarded as independent of the size of the surface-element. This quotient, which may be denoted by $\mathbf{P}$, is called the *tension* acting on the surface-element. It must, however, be remembered that the direction and magnitude of this vector will, in general, depend on the *orientation of the surface-element df*.

Hence, if ϱ be the density of a volume-element $d\tau$, and $\mathbf{K}$ be the external force acting on unit-volume (*i.e.*, the density of external force), we have the relation

$$(1) \qquad \frac{d}{dt}\int \varrho\mathbf{v}\,d\tau = \int \varrho\mathbf{K}\,d\tau + \int \mathbf{P}\,df.$$

§ 45. Stress.

As we have already seen, the effective tension on a surface-element within a deformable body depends not only on the position, but also on the *direction* of the surface-element. Let us consider any point within a deformable body and, constructing a coordinate system about it, denote by $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$ the tensions acting on a surface-element situated at this point; we shall make one of the three fundamental vectors of the coordinate system coincide with the direction of the normal to the surface pointing outwards. Thus, for example, $\mathbf{P}_1$ is the tension acting on a surface-element at right angles to the x-axis, when the sense of the outwardly directed normal is that in which the x-coordinates increase.

Now it is obvious that the force exerted by the particles to the right of a surface-element on the particles to its left is equal and opposite to the force exerted by the particles to the left on those to the right. Hence, quite generally, if the tension experienced by a surface-element with the normal $(+\,\mathbf{n})$ be denoted by $(+\,\mathbf{P})$, the tension acting on a surface-element with the normal $(-\,\mathbf{n})$ in the same position— or very close to it—must be equal to $(-\,\mathbf{P})$. The tension

exerted on a surface-element at right angles to the x-axis, and with a normal opposed to the fundamental vector **i**, is thus equal to $(-\mathbf{P}_1)$.

Let us now draw through the point O', where we wish to investigate the tensional forces, three straight lines parallel to the coordinate axes, and let us then construct a surface, with an *outwardly* directed normal **n**, in such a way that the point at which it intersects the three straight lines form with the point O' the four corners of an *infinitesimal tetrahedron* (Fig. 33). We shall regard the surface-element df, which is at right angles to **n** as the fundamental surface, and
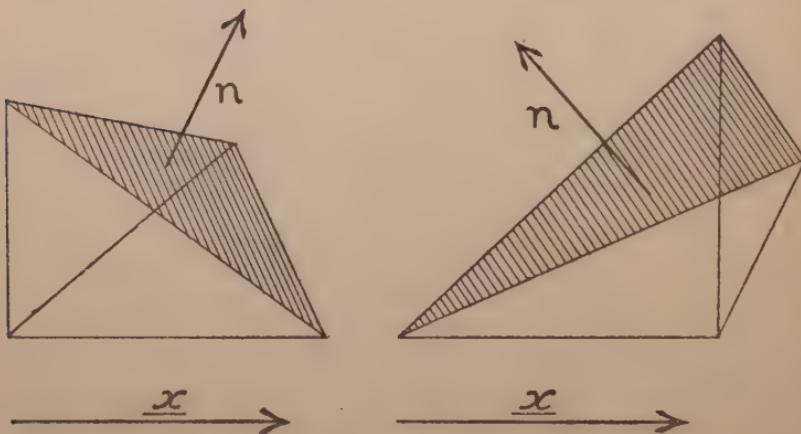
Fig. 33.

denote by df_1, df_2, df_3, the three other surfaces normal to the x-, y-, z-axes. Since

(1) $$df_1 = df \cos (\mathbf{n}, x),$$

The force acting on the surface df_1 must be equal to $\mathbf{P}_1 \cos (\mathbf{n}, x)\, df$. As regards its sign, it is easy to see that, when the angle $(\mathbf{n}, x)$ is acute, and thus $\cos (\mathbf{n}, x)$ positive, the normal to df_1 is equal to $(-\mathbf{i})$ (Fig. 33, left). On the other hand, when the angle $(\mathbf{n}, x)$ is obtuse, and thus $\cos (\mathbf{n}, x)$ negative, the normal is equal to $(+\mathbf{i})$ (Fig. 33, right). In either case the force exerted on the surface-element df_1 may therefore be put equal to

$$-\mathbf{P}_1 \cos (\mathbf{n}, x)\, df_1.$$

Since analogous relations hold for the surfaces df_2 and df_3 (normal to the y- and z-axes respectively), we find that

the total force acting on the surface of the tetrahedron (which we shall call $\mathbf{S}\,df$) is given by

(2) $\mathbf{S}\,df = \{- \mathbf{P}_1 \cos(n, x) - \mathbf{P}_2 \cos(n, y) - \mathbf{P}_3 \cos(n, z) + \mathbf{P}\}df.$

Applying § 44 eqn. (1) to the infinitesimal tetrahedron by referring the integral to its volume $(d\tau)$ and its surface, we have the relation

(3) $$\frac{d}{dt}(\varrho\mathbf{v})\,d\tau = \varrho\mathbf{K}\,d\tau + \mathbf{S}\,df.$$

Denoting by dh the height of the tetrahedron, in a direction at right angles to the surface-element df so that

(4) $$d\tau = \frac{1}{3}df \cdot dh,$$

and dividing eqn.(3) throughout by $d\tau$, we find

(5) $$\frac{d}{dt}(\varrho\mathbf{v}) = \varrho\mathbf{K} + \frac{3}{dh}\mathbf{S}.$$

Now, if we exclude infinite accelerations, infinite densities and infinite forces, neither the left-hand side of this equation nor $\varrho\mathbf{K}$ can be infinite : hence the second term on the right-hand side also cannot be infinite.   Since, however, for infinitesimal values of dh, the reciprocal $1/dh$ must become infinitely large, this possibility is only excluded when $\mathbf{S}$ is equal to zero.   Hence, from eqn. (2) we obtain the important relation

(6) $\mathbf{P} = \mathbf{P}_1 \cos(n, x) + \mathbf{P}_2 \cos(n, y) + \mathbf{P}_3 \cos(n, z).$

Denoting the components of the vector $\mathbf{P}_1$ by p_{xx}, p_{yx}, p_{zx}, those of the vector $\mathbf{P}_2$ by p_{xy}, p_{yy}, p_{zy}, and those of the vector $\mathbf{P}_3$ by p_{xz}, p_{yz}, p_{zz}, we may write eqn. (6) analytically in the form of the following triplet of equations :—

(7) $\begin{cases} P_x = p_{xx} \cos(n, x) + p_{xy} \cos(n, y) + p_{xz} \cos(n, z), \\ P_y = p_{yx} \cos(n, x) + p_{yy} \cos(n, y) + p_{yz} \cos(n, z), \\ P_z = p_{zx} \cos(n, x) + p_{zy} \cos(n, y) + p_{zz} \cos(n, z). \end{cases}$

The three cosines, however, are the components of the unit-vector $\mathbf{n}$, and eqns. (7) show us that the vector $\mathbf{P}$ is a *linear vector function* of the vector $\mathbf{n}$.   Hence, in accordance with § 26 eqns. (12), the nine quantities p_{xx}, p_{xy}, *etc*., repre-

sent the *components of a tensor*.   This tensor is known as the *stress tensor*, and we see at once from § 36 eqn. (5) that the tension is simply its *normal component*.   According to eqns. (7) and § 36 eqn. (7), therefore, the surface-integral of the tension is equal to the volume-integral of the vector divergence of the stress, or

$$(8) \qquad \int \mathbf{P} \, df = \int \operatorname{div} p \, d\tau.$$

It may readily be shown that the stress tensor is symmetrical.   If $\mathbf{r}$ be the radius-vector drawn from the origin, we have by eqn. (8)

$$(9) \qquad \int [\mathbf{r}\mathbf{P}] \, df = \int [\mathbf{r} \operatorname{div} p] \, d\tau,$$

or, written analytically,

$$(10) \qquad \int (y \, P_z - z P_y) \, df = \int (y \operatorname{div} p_z - z \operatorname{div}_y p) \, d\tau,$$

with two other analogous equations.   But, by eqns. (7)

$$(11) \quad y \, P_z = y \, \{ p_{zx} \cos (\mathbf{n}, x) + p_{zy} \cos (\mathbf{n}, y) + p_{zz} \cos (\mathbf{n}, z) \},$$

whence, in accordance with § 33 eqn. (3),

$$(12) \quad \int y P_z \, df = \int \left\{ \frac{\partial}{\partial x} (y p_{zx}) + \frac{\partial}{\partial y} (y p_{zy}) + \frac{\partial}{\partial z} (y p_{zz}) \right\} d\tau.$$

Now

$$(13) \qquad \begin{cases} \dfrac{\partial}{\partial x} (y p_{zx}) = y \dfrac{\partial p_{zx}}{\partial x}, \; \dfrac{\partial}{\partial y} (y p_{zy}) = p_{zy} + y \dfrac{\partial p_{zy}}{\partial y}, \\[2mm] \qquad\quad \dfrac{\partial}{\partial z} (y p_{zz}) = y \dfrac{\partial p_{zz}}{\partial z}. \end{cases}$$

Hence

$$(14) \qquad \begin{cases} \displaystyle \int y P \; df = \int \left\{ y \left(\dfrac{\partial p_{zx}}{\partial x} + \dfrac{\partial p_{zy}}{\partial y} + \dfrac{\partial p_{zz}}{\partial z} \right) + p_{zy} \right\} d\tau \\[3mm] \qquad\quad = \displaystyle \int (y \operatorname{div}_z p + p_{zy}) \, d\tau. \end{cases}$$

and similarly

$$(15) \qquad \int z \, P_y \, df = \int (z \operatorname{div}_y) \, p + p_{yz}) \, d\tau.$$

Introducing these values into eqn. (10), we obtain the relation

$$(16) \qquad \int (p_{zy} - p_{yz}) \, d\tau = 0.$$

Since this relation must be satisfied for any volume, even if infinitesimal, we must always have

$$(17) \qquad p_{zy} = p_{yz} \, ;$$

i.e., the stress tensor is a *symmetrical tensor*.

The tensor components of the first kind are called the *normal stresses*, and those of the second kind the *tangential stresses*. The principal values of the stress tensor are called the *principal stresses*, and the planes which are at right angles to the principal axes of the stress tensor are called the *principal planes*. In consequence of the symmetry of the stress tensor, the tangential stresses relative to a system of principal axes must vanish. Hence, if we construct within a continuously distributed mass a small *parallelopiped* whose surfaces are formed by *principal planes*, *no tangential stresses* whatsoever act upon this parallelopiped.

§ 46. The Dynamics of the Deformable Body.

The conditions holding within a continuously distributed mass may be represented by scalar and vector quantities which, in so far as they characterize properties whose *material carriers are moving particles*, appear as *functions both of the place and of the time*. If we suppose, for example, some property of a certain moving particle to be represented by a scalar S of this kind, the total variation in this scalar, during an interval of time dt, is given by the relation

$$(1) \qquad dS = \frac{\partial S}{\partial t} \, dt + \frac{\partial S}{\partial x} \, dx + \frac{\partial S}{\partial y} \, dy + \frac{\partial S}{\partial z} \, dz \, .$$

In this expression dx, dy, dz are the components of the element of path described in the element of time dt by the particle, moving with a velocity $\mathbf{v}$, which forms the material carrier of the property denoted by the scalar S. Thus dx is equal to $v_x \, dt$, and so

$$\frac{dS}{dt} = \frac{\partial S}{\partial t} + \frac{\partial S}{\partial x} \, v_x + \frac{\partial S}{\partial y} \, v_y + \frac{\partial S}{\partial z} \, v_z \, ,$$

or,

$$(2) \qquad \frac{dS}{dt} = \frac{\partial S}{\partial t} + \mathbf{v} \operatorname{grad} S.$$

In like manner, when we have to do with a vector $\mathbf{A}$ representing a property of a moving particle, instead of with a scalar, we find

$$\frac{dA_x}{dt} = \frac{\partial A_x}{\partial t} + \mathbf{v} \operatorname{grad} A_x,$$

or [in accordance with § 32 eqn. (14)],[1]

$$(3) \qquad \frac{d\mathbf{A}}{dt} = \frac{\partial \mathbf{A}}{\partial t} + (\mathbf{v} \operatorname{grad}) \mathbf{A}.$$

When, in particular, the vector $\mathbf{A}$ is the velocity of the moving particle, the total time-derivative of the velocity represents the *acceleration* of the latter, for which we consequently have the following relation :—

$$(4) \qquad \mathbf{b} = \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \operatorname{grad}) \mathbf{v}.$$

Multiplying now the acceleration vector by the mass of the particle in question, and taking the sum over all the particles contained in a delimited volume, we thereby obtain the volume-integral of the product of the mass density (ϱ) and the acceleration. In accordance with Newton's Second Law of Motion, this integral must be equal to the volume-integral of the density of internal force, in the absence of external forces.[2] Thus we have the relation

$$(5) \qquad \varrho\mathbf{b} = \mathbf{q} = \operatorname{div} p,$$

where each of the quantities is to be regarded as a function both of the place and of the time.

When an *external force* proportional to the continuously distributed mass is also acting upon the latter, we have

$$(6) \qquad \mathbf{b} = \frac{\mathbf{q}}{\varrho} + \mathbf{K},$$

where $\mathbf{K}$ denotes the external force acting on unit-volume,

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[1] The partial time-derivatives describe the so-called *local variation* of the property represented by the scalar S or the vector $\mathbf{A}$; while the total time-derivatives describe the so-called *material variation*. That is to say, in the first case the variation of the condition at one and the same place is described ; in the second, that in one and the same particle.

[2] By density of internal force we mean the quotient of the total internal force acting across the bounding surface of the volume and the volume, *i.e.*, the internal force referred to unit-volume; according to § 45 eqn. (8), this is equal to the vector divergence of the stress.

and hence, in accordance with eqns. (4) and (5), we obtain the *equation of motion of a deformable mass* in the form

$$(7) \qquad \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \text{ grad}) \, \mathbf{v} = \frac{1}{\varrho} \, \text{div} \, p + \mathbf{K} \, .$$

The above equation of motion is supplemented by a further fundamental relation arising from the Principle of the *Conservation of Mass*, according to which mass can neither be created nor destroyed. If we consider a delimited volume, the excess of the mass which flows out of the volume through its surface over that flowing in, during any interval of time, must be equal to the diminution in the mass contained in the volume during the same interval. Now the excess of the out-flowing mass over the in-flowing is equal to dt times the surface-integral of $\varrho v_n \, df$ (for, by definition, the normal to the surface points *outwards*, and v_n may be either positive or negative). It follows from Gauss's theorem that the excess of the out-flowing over the in-flowing mass is equal to

$$(8) \qquad dt \int \text{div} \, (\varrho \mathbf{v}) \, d\tau \, ,$$

On the other hand, the diminution in the mass contained in the volume during the interval dt is given by the expression

$$(9) \qquad - \, dt \int \frac{\partial \varrho}{\partial t} \, d\tau \, .$$

In accordance with the Principle of the Conservation of Mass, expressions (8) and (9) must be equal, and, moreover, for any arbitrary volume. We thus obtain the important relation

$$(10) \qquad \text{div} \, (\varrho \mathbf{v}) + \frac{\partial \varrho}{\partial t} = 0 \, ,$$

which is generally known as the *equation of continuity*, and which constitutes the vectorial analytical expression of the *Principle of the Conservation of Mass*.

The *motion of a rigid mass* must obviously form a *special case* of the motion of a continuously distributed mass. In accordance with § 13 eqn. (13), the former may be described by the equation

$$(11) \qquad \mathbf{v} = \mathbf{v}_t + [\mathbf{wr}] \, ;$$

where $\mathbf{v}_t$ is the translatory velocity ; $\mathbf{w}$ the angular velocity with which a coordinate system rigidly connected with the mass rotates ; and $\mathbf{r}$ the directed line drawn from the coordinate origin to the place under consideration within

the mass, at which the value of the velocity vector is **v**. Now, if we form the rotation of the velocity for a rigid mass, we find by eqn. (11) that

$$(12) \qquad \text{rot}_x \, \mathbf{v} = \frac{\partial}{\partial y} \, (w_x y - w_y x) - \frac{\partial}{\partial z} \, (w_z x - w_x z),$$

seeing that the components of **r** are the same as the coordinates, and that $\mathbf{v}_t$ is independent of position. But the vector **w** is independent of the coordinates, and the partial differentiation of one coordinate with respect to another gives either unity or zero. Hence the right-hand side of eqn. (12) is simply equal to $2w_x$; and thus quite generally for a rigid mass

$$(13) \qquad\qquad\qquad \text{rot} \, \mathbf{v} = 2\mathbf{w}.$$

Within a rigid mass the rotation of the velocity is everywhere equal to twice the angular velocity; and this relation explains why the term " rotation " has been applied to this particular differential operation in vector analysis. The reason for the apparently arbitrary stipulation made in § 32 eqn. (11) as to the sign of the " rotation " is now also evident, for the rotation of the velocity is thereby given the same sign as the angular velocity.

Let us now consider a small region within an entirely arbitrary (and hence not necessarily rigid) mass. It follows from § 32 eqn. (19) that the motion of this region can be described by the formula

$$(14) \qquad\qquad \mathbf{v} = \mathbf{v}_t + \frac{1}{2} \, [\text{rot} \, \mathbf{v}, \, \mathbf{r}] + \mathbf{v}',$$

where **v**' is a linear function of the vector **r** and the dilatation of **v**. Comparing eqns. (14) and (11), we thus see that the motion of a small region may be regarded as the *superposition of three motions*. The first of these is a *translation* of the region; the second is a *rotation* in which the region, just as a rigid body, rotates through a segment; while the third motion occurs at every point in the region with a velocity **v**' which depends on position.

It is this third motion which distinguishes a non-rigid from a rigid body : it is known as the *deformation* (or *strain*), the word " strain " (just like the word " displacement ") signifying both the movement itself and its effect, *viz.*, the consequent change in position of the particles. A movement of a continuously distributed mass, when both translation and rotation are absent, is called a pure deformation.

We will now turn our attention to a particle which, at

the beginning of an element of time, is located at a point (x, y, z), and which, at the end of the element of time, has been displaced to a point $(x + \xi, y + \eta, z + \zeta)$ in consequence of a small, pure strain. The components ξ, η, ζ determine the vector **a** of the *small displacement* experienced by the particle, as a result of the pure strain. But the vector **a** is equal to the product of the vector **v′** and the time-element. Since therefore, as was stated above, **v′** is a linear function of **r** and the dilatation of **v**, **a** must similarly be a linear function of **r** and the dilatation of **a**. In accordance with the definition of the dilatation of a vector [§ 32 eqn. (12)] we thus have the equation

$$(15) \qquad \xi = \frac{\partial \xi}{\partial x} x + \frac{1}{2} \left(\frac{\partial \xi}{\partial y} + \frac{\partial \eta}{\partial x} \right) y + \frac{1}{2} \left(\frac{\partial \xi}{\partial z} + \frac{\partial \zeta}{\partial x} \right) z,$$

with two analogous expressions for η and ζ. Just as in § 35, we may regard the partial derivatives of ξ, η, ζ with respect to the coordinates as *constant*, on account of the small size of the region.

In order to recognize the physical significance of the *dilatation of the displacement*, we will first assume that all the nine components of the dilatation vanish, with the exception of the x-x-component. Then, by eqn. (15),

$$(16) \qquad \xi = \frac{\partial \xi}{\partial x} x, \quad \eta = 0, \quad \zeta = 0.$$

Therefore, in this particular instance, the strain consists of an increase or diminution of the x-coordinates of all the particles by an amount which is *proportional* to the coordinate itself (on account of the constancy of the partial derivative). Thus the whole region is extended or contracted in the direction of x. A strain of this kind is called an *extension* in the direction of x, whereby the extension may be either positive or, for a contraction, negative.

We will now further investigate the significance of the tensor components of the second kind. To do this we shall assume that all components of the dilatation vanish, excepting the x-y- and the y-x-components which, in consequence of the symmetry of the tensor, are equal. We then have, by eqn. (15),

$$(17) \quad \xi = \frac{1}{2} \left(\frac{\partial \xi}{\partial y} + \frac{\partial \eta}{\partial x} \right) y, \quad \eta = \frac{1}{2} \left(\frac{\partial \eta}{\partial x} + \frac{\partial \xi}{\partial y} \right) x, \quad \zeta = 0,$$

where again owing to the smallness of the region, as previously stated, the two identical expressions in brackets are to be regarded as constant throughout the region. In the

first place, therefore, all particles experience a displacement
in the direction of x by an amount proportional to the y-co-
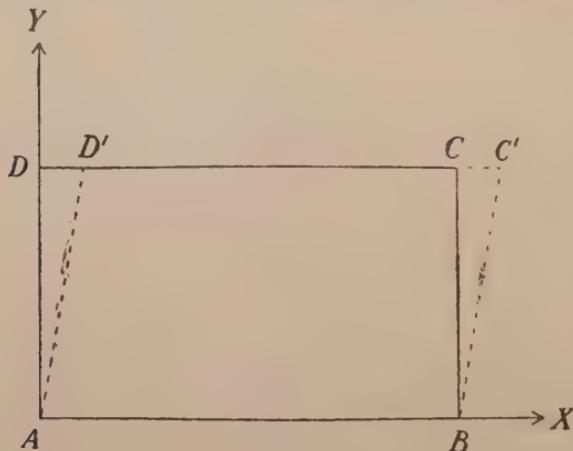ordinate. In Fig. 34, for example, a displacement of this

Fig. 34.

sort will transform the rectangle $ABCD$ into the rhomboid
$ABC'D'$. It follows from eqn. (17), however, that there is
superimposed upon this displacement a second similar one,
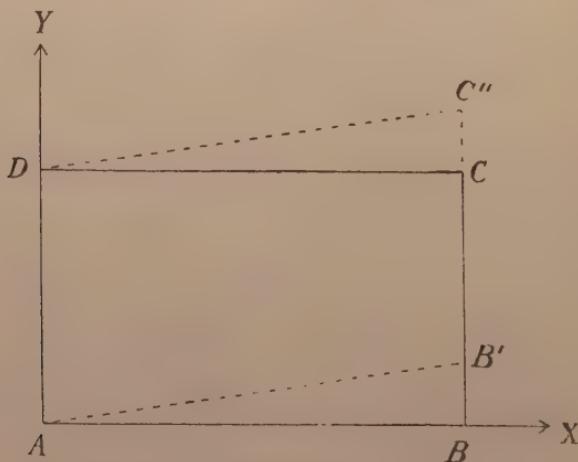in which all particles experience a displacement in the

Fig. 35.

direction of y which is proportional to the x-coordinate. If
it were only a question of this displacement, the rectangle
$ABCD$ (Fig. 35) would be transformed into the rhomboid

$AB'C''D$. Hence the total effect of the strain described by
eqn. (17) consists in the transformation of the rectangle
$ABCD$ (Fig. 36) into the quadrilateral $AB'C'''D'$, where the
displacement CC''' represents the geometrical sum of the
displacements CC' and CC''. (It is, however, always pre-
sumed that the displacements are only very small.)   A
strain of the kind described by eqn. (17) is called a *shear* in
the x-y-plane.

With reference to any given coordinate system a deforma-
tion thus appears as an extension or contraction along the
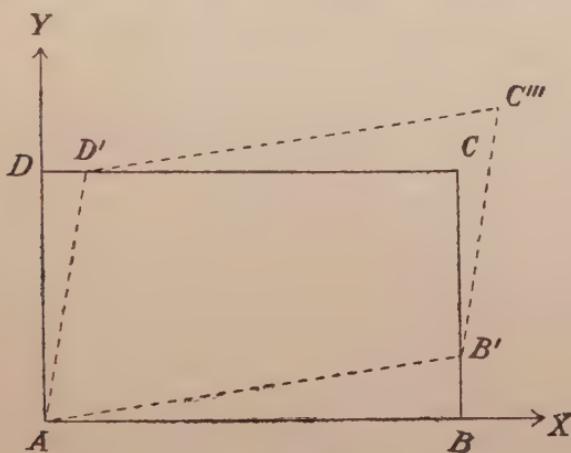
FIG. 36.

coordinate axes, and as a shear in the coordinate planes.   If,
however, the coordinate system be, in particular, so con-
structed that its axes coincide with the *principal axes of
dilatation*, the strain appears as a *simple extension* along the
coordinate axes.   When we consider the interior of a deform-
able body, it is only the strain as such that has any signi-
ficance independent of the coordinate system, whereas the
division into extension and shear simply depends on the
orientation of the coordinate system used.   As a matter of
fact, if we suppose $ABCD$ in Fig. 37 to be a square cut out
of a deformable body, then if it be transformed as a result
of deformation into the rectangle $AB'C'D'$, the strain appears
as an extension.   If, however, we denote the middle points
of the sides of the square by $EFGH$ (Fig. 38), the same
deformation changes the square $EFGH$ into a rhombus, and
this transformation would be interpreted as a shear.

Let us now suppose a small rectangular parallelopiped to

be cut out within a deformable body, its sides being parallel to the principal axes of dilatation and of lengths a, b, and c respectively. Then the volume before the strain is

$$V = abc \; ;$$

while, after the strain, it becomes

$$V' = \left(a + \frac{\partial \xi}{\partial x} a \right) \; \left(b + \frac{\partial \eta}{\partial y} b \right) \; \left(c + \frac{\partial \zeta}{\partial z} c \right).$$

As we are only considering infinitesimal strains, those terms in the triple product in which two or three partial deriva-
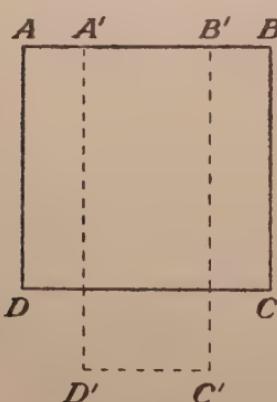
Fig. 37.
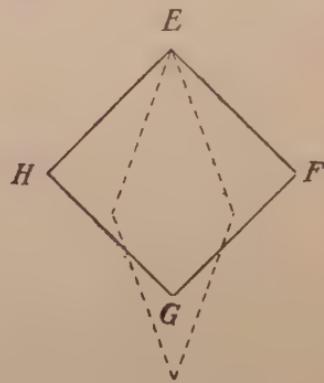
Fig. 38.

tives appear multiplied together may be neglected. We may thus put

$$(22) \qquad V' = abc \left(1 + \frac{\partial \xi}{\partial x} + \frac{\partial \eta}{\partial y} + \frac{\partial \zeta}{\partial z} \right),$$

or,

$$(23) \qquad \frac{V' - V}{V} = \operatorname{div} \mathbf{a}.$$

Therefore the *divergence of the displacement vector* represents the increase of volume per unit volume resulting from the strain, or the so-called *volume dilatation*.

§ 47. The Ideal Fluid.

An *ideal fluid* may be *defined* as a *deformable body* which possesses *no definite shape*, so that *alterations in form* unaccompanied by changes in volume *require no work*. If we imagine any given continuously distributed mass to be divided up into a number of very small cubes (whose planes may be parallel to the coordinate planes of a system), we

may regard a change in the shape of the body which is not accompanied by a change in volume, to be produced by a displacement of these elementary cubes relative to one another, *e.g.*, just as with a heap of sand grains. As these displacements occur parallel to the surfaces of the cubes, and thus perpendicular to the normal stresses, the work done in the course of such a change of shape of a deformable body can only be due to the tangential stresses.

Now we have stated in the definition of an ideal fluid that no work shall be done during an alteration of form ; and this can only be the case if, in a deformable body of this kind, the *whole of the tangential stresses vanish*, relative to *any* coordinate system whatsoever. Since any direction then represents a principal axis of the stress tensor (according to § 27), the ellipsoid of stress must become a *sphere* in the case of an ideal fluid. Moreover, the three tensor components of the first kind, with respect to any arbitrary coordinate system, must therefore have the same value. The latter may be denoted by $-p$,[1] where by p we are to understand the *fluid pressure* (which is opposed to the stress taken as being positive). Hence, in an ideal fluid, the *pressure is entirely independent of direction ;*[2] and we always have

$$(1) \qquad \begin{cases} p_{xx} = p_{yy} = p_{zz} = -p\,, \\ p_{xy} = p_{yx} = p_{yz} = p_{zy} = p_{zx} = p_{xz} = 0\,. \end{cases}$$

In accordance with the relations connecting the vector divergence of a tensor with the tensor itself [§ 36 eqn. (4)], we find that, in an ideal fluid, the *density of internal force* (which is the vector divergence of the stress) is equal to the *negative gradient of the pressure*, *i.e.*, that

$$(2) \qquad \mathfrak{q} = -\operatorname{grad} p.$$

Taking into account § 46 eqn. (7), we thus obtain as the *fundamental equation of hydrodynamics*

$$(3) \qquad \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\operatorname{grad})\,\mathbf{v} = -\frac{1}{\varrho}\operatorname{grad} p + \mathbf{K}.$$

This equation, when resolved into the three corresponding analytical expressions, gives the so-called *Euler Equations* (named after their discoverer).[3]

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[1] It must be borne in mind that the letter p has a different signification in this section (§ 47) from that in the other sections.

[2] A liquid for which this is not the case is termed *viscous.*

[3] These therefore run :

$$\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z} = -\frac{1}{\varrho}\frac{\partial p}{\partial x} + K_x,\ etc.$$

When, in particular, the ideal fluid is *incompressible* (as, for instance, is approximately the case with water, but not with gases), the density is then constant, and the *equation of continuity* [§ 46 eqn. (10)] hence assumes the simple form

(4)   div **v** = 0 .

Consequently, eqn. (4) forms a fourth relation to be added to the three expressions resulting from the analytical resolution of eqn. (3). If the constant density of the fluid be known, there remain four unknown quantities connected by the four equations thus obtained, *viz.*, the three components of the velocity and the pressure. All four unknowns are, however, functions both of the place and of the time.[4]

By means of § 32 eqn. (30) we are able to undertake an important transformation of the fundamental equation of hydrodynamics ; [5] for we have

$$(5) \qquad (\mathbf{v}\ \mathrm{grad})\ \mathbf{v} = \mathrm{grad}\left(\frac{v^2}{2}\right) - [\mathbf{v},\ \mathrm{rot}\ \mathbf{v}].$$

Now, in a fluid, half the rotation of the velocity is known as the *vortex velocity* [in accordance with § 46 eqn. (13)] ; we shall denote it by **w**. We shall further make the special assumption that the fluid is *incompressible*, and that the *volume force* **K** possesses a *potential* Φ. Substituting for the left-hand side of eqn. (5) the value given by eqn. (3), we thus obtain the relation

$$(6) \qquad - \mathrm{grad}\left(\frac{p}{\varrho} + \Phi + \frac{v^2}{2}\right) = \frac{\partial \mathbf{v}}{\partial t} - 2\,[\mathbf{vw}].$$

Again, since the rotation of the gradient of a scalar must always vanish [by § 32 eqn. (23)], it follows that the rotation of the right-hand side of eqn. (6) must also vanish, *i.e.*, that

$$(7) \qquad \mathrm{rot}\ \frac{\partial \mathbf{v}}{\partial t} = 2\ \mathrm{rot}\ [\mathbf{vw}].$$

Now

$$(8) \qquad \mathrm{rot}\ \frac{\partial \mathbf{v}}{\partial t} = 2\,\frac{\partial \mathbf{w}}{\partial t};$$

and, by § 32 eqn. (29),

$$(9) \qquad \left\{ \begin{aligned} \mathrm{rot}\ [\mathbf{vw}] = \mathbf{v}\ \mathrm{div}\ \mathbf{w} - \mathbf{w}\ \mathrm{div}\ \mathbf{v} + (\mathbf{w}\ \mathrm{grad})\ \mathbf{v} \\ - (\mathbf{v}\ \mathrm{grad})\ \mathbf{w} . \end{aligned} \right.$$

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[4] In the case of gases, eqn. (4) is replaced by Boyle's Law, or by the more complicated equation of state.

[5] This transformation is usually called *Weber's Transformation*, after its discoverer, Heinrich Weber. By means of it, eqn. (6) may be deduced from eqn. (3).

We see from eqn. (4) that the second term on the right-hand side of this equation vanishes. Moreover the first term also vanishes, inasmuch as $\mathbf{w}$ is equal to half the rotation of $\mathbf{v}$, and the divergence of the rotation of any arbitrary vector (thus of $\mathbf{v}$ itself) vanishes, in accordance with § 32 eqn. (24). Eqn. (7) therefore becomes

$$(10) \qquad 2\,\frac{\partial \mathbf{w}}{\partial t} = 2\,(\mathbf{w}\ \mathrm{grad})\,\mathbf{v} - 2\,(\mathbf{v}\ \mathrm{grad})\,\mathbf{w}\,.$$

Again, by § 46 eqn. (3),

$$(11) \qquad \frac{d\mathbf{w}}{dt} = \frac{\partial \mathbf{w}}{\partial t} + (\mathbf{v}\ \mathrm{grad})\,\mathbf{w}\,.$$

Hence eqn. (10) may be reduced to the simple form

$$(12) \qquad \frac{d\mathbf{w}}{dt} = (\mathbf{w}\ \mathrm{grad})\,\mathbf{v}\,.$$

This equation was first established by *Helmholtz* (1858),[6] and was used by him for the derivation of important theorems on *vortex motion*. By vortex motion we understand the movement of a region within a fluid when it rotates as a rigid body. We speak of a fluid particle as being in vortex motion when the vector $\mathbf{w}$ (whose material carrier it is) has a value other than zero. We see at once from eqn. (12) that a fluid particle, which is not originally in vortex motion, cannot begin to describe a vortex motion in the course of time. Conversely, it also follows from eqn. (12) that a particle that is once in vortex motion can never cease to be so. Hence, in an ideal, incompressible fluid, upon which only such external forces act as possess a potential, a *vortex motion* can *neither develop nor*, if once present, can it *cease*.

The theorems derived quite generally in § 34 for source-free fields, moreover, hold good, of course, for vortex motion in particular. Also, vortex filaments in a fluid can neither begin nor end within it, and the product of the cross-section of a filament and the vortex velocity must be constant along the filament.[7]

If there are no vortex motions present in a fluid, so that the rotation of the velocity vanishes throughout, it follows from § 32 eqn. (23) that the velocity can be represented as

[6] Analytically written, Helmholtz's equation runs :

$$\frac{dw_x}{dt} = w_x\,\frac{\partial v_x}{\partial x} + w_y\,\frac{\partial v_x}{\partial y} + w_z\,\frac{\partial v_x}{\partial z}, \quad etc.$$

[7] We shall not here further discuss other theorems derived by Helmholtz from his equation.

the negative gradient of a scalar which is called the *potential of the velocity*. An irrotational motion of a fluid is called a *flow* (or streaming). When, in particular, the value of the velocity vector remains constant at a certain place, so that

$$(13) \qquad \frac{\partial \mathbf{v}}{\partial t} = 0 \,,$$

we speak of a *steady flow*. In that case the expression in brackets on the left-hand side of eqn. (6) becomes constant. Now the value assumed by p when $v = 0$ is termed the *hydrostatic pressure* (p_0), while p itself is called the *hydraulic pressure*. Hence, in the case of a steady flow, we find from eqn. (6) that

$$(14) \qquad p = p_0 - \frac{\varrho v^2}{2} \,.$$

Therefore the hydraulic pressure is always smaller than the hydrostatic. It becomes negative as soon as a " critical " velocity is exceeded, which can never be greater than the square root of the air pressure multiplied by $2/\varrho$.[8]

Since, by eqn. (4), the divergence of the velocity vanishes in an incompressible fluid, the *velocity vector in an incompressible fluid* is therefore *solenoidal* (in the sense of § 34). The vector lines of the velocity are called *stream-lines ;* and, according to § 34, they can neither begin nor end within the fluid. If we consider the flow of a fluid within a tube, we may regard the latter as a vector tube of the velocity, seeing that the flow takes place parallel to its walls. From § 34 eqn. (3) follows at once the important theorem that the *velocity of flow* within a tube is *inversely proportional to the cross-sectional area of the tube*.

§ 48. Elastic Media.

The *relation between stress and strain* in actual bodies can be described by means of a law which was first established empirically by *Hooke* (1678), and later extended by *Navier* (1822). Hooke's Law states that every tension produces an *extension in the direction of the tension and proportional to it*. Defining the reciprocal of the proportional factor as the *modulus of elasticity*,[1] and denoting it by E, we have

$$(1) \qquad \frac{\partial \xi}{\partial x} = \frac{p_1}{E} \,,$$

[8] In the case of water this critical velocity, at which, for example, a jet of water breaks up, amounts to some 14 metres per second.

[1] Or as *Young's Modulus.*

where p_1 is the value of the tension, and its direction is that of the x-axis.

Navier's Law supplements that of Hooke by stating that every extension is accompanied by a *transverse contraction,* also proportional to the extension, in all directions perpendicular to the latter. If k be the proportional factor, we then have

$$(2) \qquad \frac{\partial \eta}{\partial y} = \frac{\partial \zeta}{\partial z} = - k \frac{p_1}{E}.$$

Experience shows that the value of the quantity k, which is known as the *coefficient of transverse contraction,*[2] lies between $1/4$ and $1/2$ for most substances.

Now a continuously distributed mass is defined as being *elastic,* when the principal axes of dilatation everywhere coincide with the principal axes of stress, and the principal stresses produce extensions and contractions in accordance with the laws of Hooke and Navier. Let t_1, t_2, t_3 be the principal values of the dilatation tensor, and p_1, p_2, p_3 the principal stresses. Then, since the strains produced by the three principal stresses are superimposed, the following relations will hold, in accordance with eqns. (1) and (2) :—

$$(3) \qquad \begin{cases} t_1 = \dfrac{p_1}{E} - \dfrac{kp_2}{E} - \dfrac{kp_3}{E}, \\[2mm] t_2 = - \dfrac{kp_1}{E} + \dfrac{p_2}{E} - \dfrac{kp_3}{E}, \\[2mm] t_3 = - \dfrac{kp_1}{E} - \dfrac{kp_2}{E} + \dfrac{p_3}{E}. \end{cases}$$

But the sum of the three principal values of the dilatation tensor (since it is equal to the divergence of the displacement) represents the *volume dilatation,* which may be denoted by Θ. Therefore, adding together the three eqns. (3), we find

$$(4) \qquad \Theta = (p_1 + p_2 + p_3) \frac{1 - 2k}{E}.$$

Now we may also write eqn. (3) in the form

$$t_1 = p_1 \frac{1 + k}{E} - \frac{k}{E} (p_1 + p_2 + p_3),$$

or, taking eqn. (4) into account,

$$(5) \qquad t_1 = p_1 \frac{1 + k}{E} - \frac{k}{1 - 2k} \Theta.$$

[2] Or as *Poisson's Ratio.*

Solving eqn. (5) in terms of p_1, and putting for short

(6) $$\frac{E}{1+k} = 2\mu, \qquad \frac{Ek}{(1+k)(1-2k)} = \lambda,$$

we find

(7) $$\begin{cases} p_1 = 2\mu\, t_1 + \lambda\,\Theta, \\ p_2 = 2\mu\, t_2 + \lambda\,\Theta, \\ p_3 = 2\mu\, t_3 + \lambda\,\Theta. \end{cases}$$

Let us now take for our coordinate system an entirely arbitrary system (x, y, z), instead of one whose axes coincide with the directions of the principal stresses. According to § 27 eqn. (5), we then have the following transformation formulæ :—

(8) $$\begin{cases} p_{xx} = \alpha_1{}^2\, p_1 + \beta_1{}^2\, p_2 + \gamma_1{}^2\, p_3, \\ t_{xx} = \alpha_1{}^2\, t_1 + \beta_1{}^2\, t_2 + \gamma_1{}^2\, t_3; \end{cases}$$

where, by § 9 eqn. (9),

(9) $$\alpha_1{}^2 + \beta_1{}^2 + \gamma_1{}^2 = 1.$$

On multiplying the three eqns. (7) by $\alpha_1{}^2$, $\beta_1{}^2$, $\gamma_1{}^2$ respectively, and adding, we find

(10) $$p_{xx} = 2\mu\, t_{xx} + \lambda\,\Theta.$$

The relations holding for the transformation of the tensor components of the second kind are [according to § 27 eqn. (5)] given by

(11) $$\begin{cases} p_{xy} = \alpha_1\alpha_2\, p_1 + \beta_1\beta_2\, p_2 + \gamma_1\gamma_2\, p_3, \\ t_{xy} = \alpha_1\alpha_2\, t_1 + \beta_1\beta_2\, t_2 + \gamma_1\gamma_2\, t_3; \end{cases}$$

where, by § 9 eqn. (10),

(12) $$\alpha_1\alpha_2 + \beta_1\beta_2 + \gamma_1\gamma_2 = 0.$$

On multiplying the three eqns. (7) by $\alpha_1\alpha_2$, $\beta_1\beta_2$, $\gamma_1\gamma_2$ respectively, and adding, we find

(13) $$p_{xy} = 2\mu\, t_{xy}.$$

Replacing the components of the dilatation tensor by their values, we consequently obtain the equations

(14) $$\begin{cases} p_{xx} = 2\mu\, \dfrac{\partial \xi}{\partial x} + \lambda\,\Theta, \\[2mm] p_{xy} = \mu\left(\dfrac{\partial \xi}{\partial y} + \dfrac{\partial \eta}{\partial x}\right), \\[2mm] p_{xz} = \mu\left(\dfrac{\partial \xi}{\partial z} + \dfrac{\partial \zeta}{\partial x}\right). \end{cases}$$

If we now proceed to form the vector divergence of the tensor, we find,

$$
(15) \quad
\begin{cases}
\dfrac{\partial p_{xx}}{\partial x} + \dfrac{\partial p_{xy}}{\partial y} + \dfrac{\partial p_{xz}}{\partial z} = \mu \left(\dfrac{\partial^2 \xi}{\partial x^2} + \dfrac{\partial^2 \xi}{\partial y^2} + \dfrac{\partial^2 \xi}{\partial z^2} \right) \\[2mm]
\qquad\qquad + \mu \dfrac{\partial}{\partial x} \left(\dfrac{\partial \xi}{\partial x} + \dfrac{\partial \eta}{\partial y} + \dfrac{\partial \zeta}{\partial z} \right) + \lambda \dfrac{\partial \Theta}{\partial x} .
\end{cases}
$$

The first expression in brackets on the right-hand side is $\nabla^2 \xi$, while the expression in the second brackets is equal to Θ. Hence

$$
(16) \qquad \mathbf{div}_x \, p = \mu \, \nabla^2 \xi + (\mu + \lambda) \, \dfrac{\partial \Theta}{\partial x} .
$$

When there is no translation or rotation of the elastic medium (or when such is disregarded), the normal position of the single particles represented by the coordinates x, y, z remains unaltered, and thus the vector (**a**) of the displacement, whose components are ξ, η, ζ, may be regarded as a function of x, y, z, and t. The components of the acceleration at any point are then given by the second partial time-derivatives $\partial^2 \xi/\partial t^2$, $\partial^2 \eta/\partial t^2$, $\partial^2 \zeta/\partial t^2$.

In accordance with § 46 eqn. (5), however, the product of the mass density and the acceleration must be equal to the vector divergence of the stress, and it therefore follows from eqn. (16) that

$$
(17) \qquad \varrho \, \dfrac{\partial^2 \xi}{\partial t^2} = \mu \, \nabla^2 \xi + (\mu + \lambda) \, \dfrac{\partial \Theta}{\partial x} ;
$$

while there are two analogous expressions for η and ζ.

Let us now differentiate these three equations partially with respect to x, y, z respectively, and then add the results together. In this way we find

$$
\varrho \, \dfrac{\partial^2}{\partial t^2} \left(\dfrac{\partial \xi}{\partial x} + \dfrac{\partial \eta}{\partial y} + \dfrac{\partial \zeta}{\partial z} \right) = \mu \, \nabla^2 \left(\dfrac{\partial \xi}{\partial x} + \dfrac{\partial \eta}{\partial y} + \dfrac{\partial \zeta}{\partial z} \right) + (\mu + \lambda) \, \nabla^2 \Theta ,
$$

or,

$$
(18) \qquad \varrho \, \dfrac{\partial^2 \Theta}{\partial t^2} = (2\mu + \lambda) \, \nabla^2 \Theta .
$$

Among elastic media we may also count liquid or gaseous *fluids*. In such bodies the tangential stresses must always vanish (by § 47), and we thus see from eqn. (14) that, from the theoretical point of view of elasticity, fluids are characterized by the disappearance of the constant μ.

§ 49. Elastic Waves.

It immediately follows from the equation just derived [§ 48 eqn. (18)], and from the considerations of § 43, that a wave-like propagation of the dilatation is possible in elastic media—*i.e.*, there may be *dilatational waves*, the value of whose *velocity* is given by

$$(1) \qquad v' = \sqrt{\frac{2\mu + \lambda}{\varrho}}.$$

It is also possible, however, for there to be *dilatation-free waves* in elastic media, as we may see from their equations of motion. For, if we put

$$(2) \qquad \Theta = 0,$$

§ 48 eqn. (17) becomes

$$(3) \qquad \varrho \, \frac{\partial^2 \mathbf{a}}{\partial t^2} = \mu \, \nabla^2 \mathbf{a},$$

where the vector $\mathbf{a}$ denotes the displacement experienced by a particle as a result of the elastic strain.

We see at once from eqn. (3) and from the results of § 43 that it is possible for the vector $\mathbf{a}$ to be propagated in plane waves. Since, however, Θ represents the divergence of $\mathbf{a}$, and inasmuch as this is equal to zero (by eqn. 2), it follows from § 43 that the *dilatation-free waves* must be *transversal*. Thus, if we choose the direction of propagation of the waves as x-axis, only the y- and z-components of the vector $\mathbf{a}$ take part in the vibrations ; and, since this vector represents the displacement from the position of rest, the particles in the dilatationless wave must move *at right angles* to the direction of propagation of the wave. For the *velocity of the dilatation-free waves* we find from eqn. (3)

$$(4) \qquad v'' = \sqrt{\frac{\mu}{\varrho}}.$$

Finally, we will investigate the direction in which the particles vibrate in the previously considered dilatational waves. For this purpose we return to § 48 eqn. (17), writing this in its vectorial form :

$$(5) \qquad \varrho \, \frac{\partial^2 \mathbf{a}}{\partial t^2} = \mu \, \nabla^2 \mathbf{a} + (\mu + \lambda) \, \mathrm{grad} \, \Theta .$$

Since the state of vibration in plane waves is the same throughout a plane perpendicular to the direction of propagation, grad Θ must have the same direction as the latter.

To avoid an unnecessary complication of our considerations, we will now suppose that the vibrations of the particles are linear, and that the direction of vibration is everywhere the same. In that case the three vectors $\mathbf{a}$, $\partial^2\mathbf{a}/\partial t^2$, $\nabla^2\mathbf{a}$ must all be similarly directed ; and consequently, in accordance with eqn. (5), the vector $\mathbf{a}$ again must have everywhere the same direction as grad Θ. It therefore follows from what we have previously said that the vector $\mathbf{a}$ vibrates in the direction of propagation of the waves. Hence, in *dilatational waves*, the particles vibrate in the direction of propagation of the waves, *i.e.*, they execute so-called *longitudinal vibrations*. As we may see from eqns. (1) and (4), the velocity of propagation of longitudinal waves is always greater than that of transversal ones.

In the case of frictionless fluids μ is always zero, as has already been mentioned in § 48. Hence *in fluids* (*e.g.*, in water or air) *only longitudinal waves* can occur. *In solid bodies both longitudinal and transversal waves* are, in general, possible. It is only in *incompressible solid bodies*, in which any volume dilatation is therefore impossible, that we find *pure transversal waves* unaccompanied by longitudinal waves.

THEORY OF POTENTIAL

§ 50. Source and Strength of Field.

When we are able to associate with any arbitrary point P in a field a *scalar* which is *inversely proportional* to the *distance* from a certain point O, we have a very simple special case of a field which is of great importance in physics. The point O is then known as the *source*, while the point P may be called the *field-point*. We may further regard the proportional factor between the scalar and the reciprocal distance as the product of a universal constant, whose value depends on the system of measurement used, and a quantity which we shall term the *strength of the source*. In the following considerations we shall simply put the constant equal to unity. Then, denoting the scalar by Ψ and the strength by g,

$$(1) \qquad \Psi = \frac{g}{r}.$$

Let the coordinates of the field-point be x_a, y_a, z_a and those of the source x_q, y_q, z_q, relative to any given coordinate system. We may then regard either the source as fixed and the field-point as variable (as was the case in § 10), or the field-point as fixed and the source as variable. In the first case we shall speak of a *gradient for a variable field-point*, denoting it by suffixing the index a to the gradient symbol. If $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are the fundamental vectors of the coordinate system, we thus have

$$(2) \qquad \operatorname{grad}_a \Psi = \mathbf{i}\, \frac{\partial \Psi}{\partial x_a} + \mathbf{j}\, \frac{\partial \Psi}{\partial y_a} + \mathbf{k}\, \frac{\partial \Psi}{\partial z_a}.$$

Again, the *gradient for a variable source*, denoted by the suffix q, will be given by

$$(3) \qquad \operatorname{grad}_q \Psi = \mathbf{i}\, \frac{\partial \Psi}{\partial x_q} + \mathbf{j}\, \frac{\partial \Psi}{\partial y_q} + \mathbf{k}\, \frac{\partial \Psi}{\partial z_q}.$$

Now, as has already been shown [§ 10 eqn. (18)], and as at

once follows from eqn. (2) for a fixed source (as was assumed in § 10),

$$(4) \qquad \operatorname{grad}_a \left(\frac{1}{r} \right) = - \frac{1}{r^2} \frac{\mathbf{r}}{r},$$

where $\mathbf{r}$ denotes the directed line joining the source to the field-point. In like manner, if for the moment we regard O as the field-point and P as the source,

$$(5) \qquad \operatorname{grad}_q \left(\frac{1}{r} \right) = - \frac{1}{r^2} \frac{\mathbf{r}'}{r},$$

where $\mathbf{r}'$ now denotes the directed line joining the (original) field-point to the (original) source. But $\mathbf{r}'$ is *opposite* and equal to $\mathbf{r}$, and hence

$$(6) \qquad \operatorname{grad}_q \left(\frac{1}{r} \right) = - \operatorname{grad}_a \left(\frac{1}{r} \right).$$

We define the negative gradient of the scalar for a variable field-point as the *strength of field*, so that the latter possesses a *potential* in the scalar (as follows from the definition of potential in § 11). Hence, if $\mathbf{F}$ be the strength of field we have

$$(7) \qquad \mathbf{F} = - \operatorname{grad}_a \Psi,$$

or, by eqns. (1) and (4),

$$(8) \qquad \mathbf{F} = \frac{g}{r^2} \frac{\mathbf{r}}{r}.$$

A simple relation holds for the *flux of the field-strength* through any elementary surface. In accordance with the definition of § 33 and with eqn. (8), the flux is equal to

$$(9) \qquad F_n \, df = \mathbf{F}\mathbf{n} \, df = \frac{g}{r^2} \frac{\mathbf{r}}{r} \, \mathbf{n} \, df,$$

where $\mathbf{n}$ is the unit-vector normal to the elementary surface, and directed outwards. The scalar product of the unit-vector $\mathbf{n}$ and the unit-vector $\mathbf{r}/r$, however, is simply the cosine of the angle between $\mathbf{r}$ and $\mathbf{n}$. We therefore have

$$(10) \qquad F_n \, df = g \frac{df \cos (\mathbf{r}, \mathbf{n})}{r^2}.$$

We may arrive at the meaning of the expression by which the strength of source g appears multiplied on the right-hand side of eqn. (10) by supposing a spherical surface to be constructed *about the source* with a radius equal to one unit of length—a so-called *unit-sphere*. We can immediately see that this expression simply represents the area

cut out of the unit-sphere by a *cone* whose vertex is the source, and whose base is the surface-element df. If we denote by $d\omega$ the elementary area cut out of the unit-sphere, $d\omega$ is the *solid angle* subtended by the surface-element df at the source (Fig. 39). We thus find
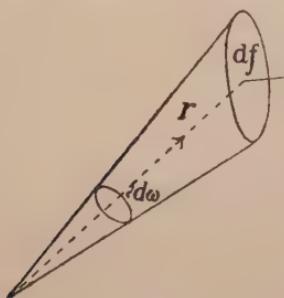

FIG. 39.

(11) $$\frac{df}{r^2}\frac{\mathbf{r}}{r}\,\mathbf{n} = d\omega\,,$$

where $d\omega$ is to be taken as *positive* or *negative* according as the continuation of the v e c t o r $\mathbf{r}$ (drawn from the source) forms an *acute* or an *obtuse angle* with the outwards-drawn normal $\mathbf{n}$ to the elementary surface. Again, taking eqns. (4) and (6) i n t o account, we obtain the following relation which is important for subsequent considerations :—

(12) $$-\,\mathbf{n}\operatorname{grad}_a\left(\frac{1}{r}\right)df = \mathbf{n}\operatorname{grad}_q\left(\frac{1}{r}\right)df = d\omega\,.$$

It follows from eqns. (9) and (11) that

(13) $$F_n\,df = g\,d\omega.$$

If we now suppose any arbitrary *surface* to be constructed around and *enclosing the source*, the integral of the solid angle taken over this surface will be equal to the total superficial area of the unit-sphere, *i.e.*, to 4π. We thus find, for a surface of this kind,

(14) $$\int F_n\,df = 4\pi g\,.$$

The *flux of the strength of field through a surface which encloses a source is equal to the strength of the source multiplied by* 4π.

When, on the other hand, the source is *not enclosed* by the surface (Fig. 40), every elementary cone drawn from the source to an element of the closed surface cuts a *second elementary surface* out of the latter. In this case the angle between the continua-

FIG. 40.

tion of $\mathbf{r}$ and $\mathbf{n}$ is obtuse for the element (df_1) nearer the source while it is acute for the other element (df_2). Hence

$$\frac{df_1}{r_1{}^2}\cos(\mathbf{n}_1,\,\mathbf{r}) + \frac{df_2}{r_2{}^2}\cos(\mathbf{n}_2,\,\mathbf{r}) = -\,d\omega + d\omega = 0.$$
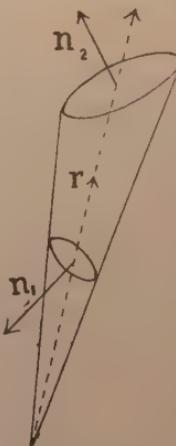
The flux of the strength of a field produced by a source through a closed surface which, however, does *not enclose* the source, is therefore *zero*.

Consequently, if we represent the vector field of the field-strength by *vector lines* in such a way that their density is everywhere proportional to the flux of the vector per unit of surface (§ 34), there must originate in every source 4π times as many vector lines as amount to the strength of the source (the latter being further multiplied by the arbitrary proportional factor). Herein is to be found the explanation of the terms "source" and "strength of source." Conversely, when the strength of source is *negative*, $4\pi g$ vector lines must obviously *terminate* in the source.

If, now, there be n fields mutually *superimposed*, each field being due to a single separate source, the potential at any field-point is given by

$$(15) \qquad \Psi = \sum_{h=1}^{h=n} \frac{g_h}{r_h}$$

Hence, for any entirely arbitrary closed surface,

$$(16) \qquad \int F_n \, df = 4\pi \sum{}' g_h,$$

the dashed summation symbol indicating that the sum is to be taken *only* over those sources which lie *within* the closed surface.

§ 51. Poisson's Equation.

Hitherto we have considered only discrete sources, but we shall now transfer our attention to a *continuous distribution of sources*. We shall define the strength of source per unit of volume simply as the *density*, denoting it by ϱ. We may thus regard every *elementary volume $d\tau$* as a source of strength $\varrho \, d\tau$, where ϱ is the value of the density in the position of the volume-element. It follows from § 50 eqn. (15) that the potential

$$(1) \qquad \Psi = \int \frac{\varrho \, d\tau}{r},$$

where r is the distance between the elementary volume and the field-point.

Now it is true that, for the field-point itself, $1/r$ is infinitely large; but nevertheless, even when ϱ, quite generally, differs from zero at the field-point, the share contributed towards the integral in eqn. (1) by the elementary volume which contains that point itself is not infinitely large. We

can easily see this by supposing a small sphere of radius a to be constructed about the field-point, within which (because it is small) ϱ may be regarded as constant. On taking the integral over this small sphere, which we imagine to be divided up into spherical shells, we find

$$\varrho \int_o^a \frac{4\pi r^2\, dr}{r} = 4\pi\varrho\, \frac{a^2}{2}\,.$$

When a becomes an infinitesimal of the first order, the share contributed towards the integral in eqn. (1) by the small sphere enclosing the field-point therefore becomes an infinitesimal of the second order.

In the case of a continuous distribution of sources § 50 eqn. (16) becomes

$$(2) \qquad\qquad \int F_n\, df = 4\pi \int \varrho\, d\tau.$$

In accordance with *Gauss's theorem*, however, the surface-integral on the left-hand side of this equation can be replaced by the volume-integral of the divergence of the strength of field ; and, since eqn. (2) must be satisfied for any *arbitrary* volume, we obtain the important general relation

$$(3) \qquad\qquad \operatorname{div} \mathbf{F} = 4\pi\varrho\,.$$

In virtue of § 50 eqn. (7) and § 32 eqn. (20), potential and density are therefore connected by the relation

$$(4) \qquad\qquad \nabla^2\Psi = -\, 4\pi\varrho\,.$$

This equation is known as *Poisson's Equation*, after its discoverer (1813). When the density vanishes at the field-point, we have

$$(5) \qquad\qquad \nabla^2\Psi = 0\,.$$

This relation, which had already been obtained by Laplace (1789) previously to Poisson, is hence called *Laplace's Equation*, and forms a special case of Poisson's Equation.

As a matter of fact, Laplace's Equation may be directly obtained with the aid of a very simple mathematical consideration. We start out from the formula

$$(6) \qquad r^2 = (x_a - x_q)^2 + (y_a - y_q)^2 + (z_a - z_q)^2,$$

the indices referring to the field-point and source. We find from eqn. (6)

$$(7) \quad \frac{\partial}{\partial x_a}\left(\frac{1}{r}\right) = -\,\frac{1}{r^2}\frac{\partial r}{\partial x_a} = -\,\frac{1}{r^2}\frac{x_a - x_q}{r} = -\,\frac{(x_a - x_q)}{r^3}\,.$$

Partially differentiating again, we have

(8) $\dfrac{\partial^2}{\partial x_a{}^2}\left(\dfrac{1}{r}\right) = -\dfrac{1}{r^3} + \dfrac{3\,(x_a - x_q)}{r^4}\dfrac{\partial r}{\partial x_a} = -\dfrac{1}{r^3} + \dfrac{3\,(x_a - x_q)^2}{r^5}.$

On forming in like manner the second partial derivatives with respect to y_a and z_a, and then adding, we obtain

$$\nabla^2_a\left(\dfrac{1}{r}\right) = -\dfrac{3}{r^3} + \dfrac{3r^2}{r^5},$$

so that

(9) $$\nabla^2_a\left(\dfrac{1}{r}\right) = 0.$$

It actually follows from this equation (which will find an application in a later investigation), on multiplying by the strength of a source, that the Laplace derivative of the potential vanishes at every field-point in the field of a single source. This obviously holds good also for a system of any number of sources, provided that the field-point itself be not a source ; and it therefore likewise holds, in the case of a continuous distribution of sources, for a field-point where the density is zero.

Since we have made no special stipulations whatever as to the scalar quantity ϱ, it follows from eqns. (1) and (4) that the equation

(10) $$\nabla^2_a \int \dfrac{S\,d\tau}{r} = -4\pi S$$

must be satisfied for any arbitrary scalar S, where the S on the right-hand side of the equation represents the value of the scalar at the field-point. We may further replace S in eqn. (10) by the three components of a vector $\mathbf{A}$; and if we then multiply the three resulting equations by the three fundamental vectors of the coordinate system and add them together, we thereby obtain the important relation

(11) $$\nabla^2_a \int \dfrac{\mathbf{A}\,d\tau}{r} = -4\pi\mathbf{A},$$

which is satisfied for any arbitrary vector.

§ 52. Surface Sources.

An important special case of a continuous distribution of sources is presented by that of a region of sources forming a shell whose thickness is so small that it may be neglected in comparison with the other dimensions of the region.   We

then speak of a *surface source,* and define the strength per
unit of surface as the *surface density.* Let the values of the
strength of field immediately on either side of the surface
source be F_1 and F_2; and let us investigate the relations
existing between these two values of the strength of field,
when the surface density (σ) is given.

To do this we resolve each of the vectors of field-strength
into two components. One of each of these dual components
shall have the direction of the outwards-drawn *normal* to
the surface, and be denoted respectively by F_1' and F_2': the
other two components are then *tangential,* and may be
denoted by F_1'' and F_2'' (Fig. 41).

We now suppose a cylinder to be constructed on either
side of an element of the surface source as a section, its
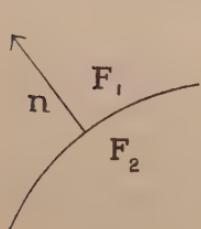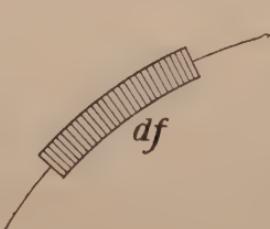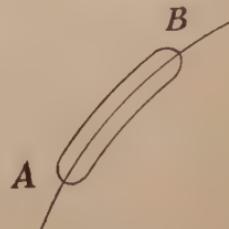
FIG. 41.   FIG. 42.   FIG. 43.

height being very small in comparison with its base (Fig. 42).
Then the normal component pointing outwards from the
cylinder, relative to the outer base, is equal to F_1'; while
the normal component pointing outwards from the cylinder,
relative to the inner base, is equal to $(-F_2')$. Since we
can make the height of the cylinder as small as we like, we
may, to any order of approximation, put the flux of field-
strength through the surface of the cylinder equal to

$$(F_1' - F_2')\, df .$$

On the other hand, this flux of the strength of field must be
equal to 4π times the strength of source contained in the
cylinder [by § 50 eqn. (16)]. The latter is, however, $\sigma\, df$;
and we thus obtain the important relation

(1) $$F_1' - F_2' = 4\pi\sigma .$$

The *normal component of the strength of field* does not traverse
a surface source continuously, but experiences a *sudden
change* which is equal to the *surface density multiplied by 4π*:
the component in the direction of the outwards-drawn

normal is greater on the outer than on the inner side, by this
amount.

Again, let us suppose a plane to be constructed cutting
the surface source at right angles, and that, within this
plane, a curve be drawn which conforms to the surface
source as closely as possible, intersecting it perpendicularly
at two neighbouring points A and B (Fig. 43). Since the
normal component of the strength of field does not become
infinite even in the surface itself [according to eqn. (1)], we
may put for the line-integral of the field-strength along
the closed curve

$$(2) \qquad \int \mathbf{F}\, d\mathbf{s} = \int_A^B \mathbf{F_1}''\, d\mathbf{s} + \int_B^A \mathbf{F_2}''\, d\mathbf{s}\,.$$

In accordance with Stokes's theorem, however, the line-
integral of $\mathbf{F}$ may be replaced by a surface-integral of the
rotation of $\mathbf{F}$. But the rotation of $\mathbf{F}$ vanishes [by § 32
eqn. (23)], seeing that $\mathbf{F}$ can be represented as the gradient
of a scalar. The left-hand side of eqn. (2) is therefore equal
to zero, so that

$$(3) \qquad \int_A^B \mathbf{F_1}''\, d\mathbf{s} = \int_A^B \mathbf{F_2}''\, d\mathbf{s}\,.$$

This equation must hold whatever the form and length of
the curve, and however the plane containing it and perpen-
dicular to the surface source be constructed. This is only
possible when, quite generally,

$$(4) \qquad\qquad \mathbf{F_1}'' = \mathbf{F_2}''\,.$$

The *tangential component of the strength of field* has therefore
the same value on either side of the surface source.

The *potential of a surface source* is given by § 50 eqn. (15)
as

$$(5) \qquad\qquad \Psi = \int \frac{\sigma\, df}{r}\,.$$

§ 53. Source-Couples and Double-Layers.

We define a *source-couple* as a system of *two rigidly con-
nected, neighbouring sources* whose *strengths are equal but
opposite.* The product of the magnitude of the strength and
the directed line drawn *from the negative to the positive
source* is defined as the *moment* of the *source-couple.* It
represents a *vector*, and may be denoted by $\mathbf{M}$.

Let us now consider any given field-point P; and let the

distances between it and the negative and positive sources be r_1 and r_2 respectively (Fig. 44). Let the line joining the negative to the positive source be $\mathbf{a}_{12}$. Then, by § 50 eqn. (15),

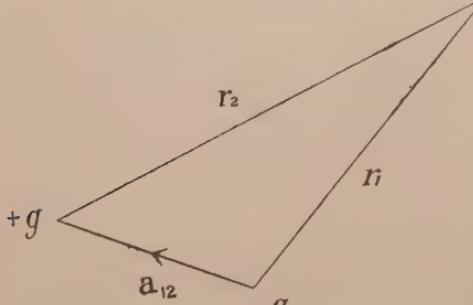$$(1) \quad \Psi = g\left(\frac{1}{r_2} - \frac{1}{r_1}\right).$$

In consequence of the small distance between the two sources (defined as neighbouring), we may [by § 10 eqn. (10)] put

Fig. 44.

$$(2) \quad \frac{1}{r_2} - \frac{1}{r_1} = \mathbf{a}_{12} \cdot \operatorname{grad}_q\left(\frac{1}{r}\right),$$

regarding, for the moment, the field-point as fixed. The product $g\mathbf{a}_{12}$, however, is by definition the moment $\mathbf{M}$: hence

$$(3) \quad \Psi = \mathbf{M}\operatorname{grad}_q\left(\frac{1}{r}\right).$$

A double surface on one face of which the surface density is equal but opposite to that on the other face, and constant throughout, may be regarded as a surface source-couple or as a so-called *double-layer*. Let a cylinder whose base is df be cut out of the double-layer, and let the distance between the two surfaces be a (Fig. 45). Let $\mathbf{n}$ be the unit-vector normal to the double-layer, drawn from the *negative to the positive surface*. Then, if $d\Psi$ be the potential, due to the element of the double-layer in question, at a field-point P at a distance r from it, we have by eqn. (3)

$$(4) \quad d\Psi = \mathbf{n}a\sigma\,df\operatorname{grad}_q\left(\frac{1}{r}\right).$$

Fig. 45.

In accordance with § 50 eqn. (12), however, we may write this

$$(5) \quad d\Psi = a\sigma\,d\omega,$$
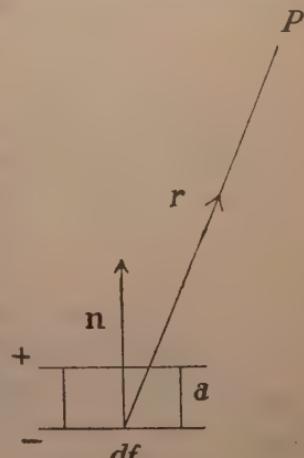
where $d\omega$ is the *solid angle* subtended *at the field-point* by the surface-element.

We must still determine when $d\omega$ is positive and when it is negative. If $\mathbf{r}$ be the line drawn from the surface-element to the field-point, we have, by § 50 eqn. (5),

$$\operatorname{grad}_q \left(\frac{1}{r} \right) = + \frac{1}{r^2} \frac{\mathbf{r}}{r}.$$

Thus $d\omega$ is positive when the scalar product of $\mathbf{r}$ and $\mathbf{n}$ is positive, *i.e.*, when the normal drawn from the negative to the positive side makes an acute angle with the vector $\mathbf{r}$. This is the case when the *field-point faces the positive side*. In the reverse instance $d\omega$ is negative.

The product $a\sigma$ gives the magnitude of the moment per unit area ; it may be termed the *density of moment* (or *strength of shell*), and be denoted by χ. By integrating eqn. (5) over the whole of the double-layer, we thus obtain the important and simple relation

(6) $$\Psi = \chi \boldsymbol{\omega} .$$

The *potential* at any field-point due to a *double-layer* is equal to the *product of the density of moment and the solid angle* subtended at that point by the double-layer. The sign will be positive or negative according as the field-point faces the positive or the negative side. Hence the potential depends *solely* upon the *bounding curve* (apart from on the density of moment), and is completely independent of the shape of the double-layer within this boundary.

§ 54. Vector Potential.

We have seen that the *scalar potential*, which we have hitherto been considering, may be referred to the function $1/r$, where r is the distance between the field-point and the source. Similarly, we may refer the so-called *vector potential* to the vector

(1) $$\mathbf{P} = \int \frac{d\mathbf{s}}{r}$$

where $d\mathbf{s}$ denotes the element of a *closed curve* which is, as it were, made up of point sources, and r is the distance of the field-point from this curve-element. In what follows we shall define, for short, the integral represented by the vector $\mathbf{P}$ as the *curve potential*.

We have by § 35 eqn. (16)

$$(2) \qquad \int \frac{d\mathbf{s}}{r} = \int \left[\mathbf{n} \operatorname{grad}_q \left(\frac{1}{r} \right) \right] df,$$

where df is the element of a surface enclosed by the curve ; for we must, of course, regard the field-point as fixed during the integration. In accordance with Stokes's theorem the sense of the normal to the surface, in the above expression, must be such that the cycle of integration along the curve appears anti-clockwise when viewed from its extremity.

Our first problem is to calculate the *rotation of the curve potential* (with, of course, a variable field-point). We find from eqns. (1) and (2)

$$(3) \qquad \operatorname{rot}_a \mathbf{P} = - \int \operatorname{rot}_a \mathbf{b} \, df,$$

putting, for short,

$$(4) \qquad \mathbf{b} = \left[\mathbf{n} \operatorname{grad}_a \left(\frac{1}{r} \right) \right],$$

and applying § 50 eqn. (6). Now

$$(5) \qquad \begin{cases} b_x = n_y \dfrac{\partial}{\partial z_a} \left(\dfrac{1}{r} \right) - n_z \dfrac{\partial}{\partial y_a} \left(\dfrac{1}{r} \right), \\[2mm] b_y = n_z \dfrac{\partial}{\partial x_a} \left(\dfrac{1}{r} \right) - n_x \dfrac{\partial}{\partial z_a} \left(\dfrac{1}{r} \right), \\[2mm] b_z = n_x \dfrac{\partial}{\partial y_a} \left(\dfrac{1}{r} \right) - n_y \dfrac{\partial}{\partial x_a} \left(\dfrac{1}{r} \right). \end{cases}$$

Again

$$\operatorname{rot}_{xa} \mathbf{b} = \frac{\partial b_z}{\partial y_a} - \frac{\partial b_y}{\partial z_a}.$$

Since, however, $\mathbf{n}$ is quite independent of the coordinates of the field-point, it follows that

$$(6) \quad \operatorname{rot}_{xa} \mathbf{b} = n_x \frac{\partial^2}{\partial y_a{}^2} \left(\frac{1}{r} \right) - n_y \frac{\partial^2}{\partial x_a \, \partial y_a} \left(\frac{1}{r} \right) - n_z \frac{\partial^2}{\partial x_a \, \partial z_a} \left(\frac{1}{r} \right)$$
$$+ n_x \frac{\partial^2}{\partial z_a{}^2} \left(\frac{1}{r} \right).$$

But, by § 51 eqn. (9),

$$\frac{\partial^2}{\partial y_a{}^2} \left(\frac{1}{r} \right) + \frac{\partial^2}{\partial z_a{}^2} \left(\frac{1}{r} \right) = - \frac{\partial^2}{\partial x_a{}^2} \left(\frac{1}{r} \right) ;$$

and we can utilize this relation to transform eqn. (6). For, seeing that the components of $\mathbf{n}$ are quite independent of the coordinates of the field-point, they can be regarded as

constant in the partial differentiation with respect to x_a, y_a, z_a. We thus have

$$(7) \begin{cases} \operatorname{rot}_{xa} \mathbf{b} = - \dfrac{\partial}{\partial x_a} \left\{ n_x \dfrac{\partial}{\partial x_a} \left(\dfrac{1}{r} \right) + n_y \dfrac{\partial}{\partial y_a} \left(\dfrac{1}{r} \right) + n_z \dfrac{\partial}{\partial z_a} \left(\dfrac{1}{r} \right) \right\} \\ \qquad = - \dfrac{\partial}{\partial x_a} \left\{ \mathbf{n} \operatorname{grad}_a \dfrac{1}{r} \right\}. \end{cases}$$

Hence, returning to eqn. (3) and taking into account § 50 eqn. (6), we find [1]

$$(8) \qquad \operatorname{rot}_{xa} \mathbf{P} = - \int \frac{\partial}{\partial x_a} \left\{ \mathbf{n} \operatorname{grad}_q \left(\frac{1}{r} \right) \right\} df.$$

Since the integration is with respect to the sources, while the partial differentiation refers to the field-point, we may interchange the order of these two operations in eqn. (8), thus finally obtaining the important relation

$$(9) \qquad \operatorname{rot}_a \int \frac{d\mathbf{s}}{r} = - \operatorname{grad}_a \int \mathbf{n} \operatorname{grad}_q \left(\frac{1}{r} \right) df.$$

In this expression, as we have already mentioned, the sense of the normal unit-vector must be such that, when viewed from its extremity, the cycle of integration appears anti-clockwise—in accordance with Stokes's theorem.

We may at once make an important application of the purely geometrical eqn. (9) to *double-layers*. It follows from § 53 eqn. (4) that the strength of field (as negative gradient of the potential), due to a double-layer whose density of moment is χ, is given by

$$(10) \qquad \mathbf{F} = - \chi \operatorname{grad}_a \int \mathbf{n} \operatorname{grad}_q \left(\frac{1}{r} \right) df,$$

the normal vector $\mathbf{n}$ being directed from the negative to the positive side. By means of eqn. (9), however, we may also write eqn. (10) in the form

$$(11) \qquad \mathbf{F} = \chi \operatorname{rot}_a \int \frac{d\mathbf{s}}{r}.$$

The *strength of field* due to a double-layer may therefore be represented, not only as the gradient of the scalar potential, but also as the *rotation of the curve potential of the boundary of the layer, multiplied by the density of moment*. The *sense* of the integration carried out along the boundary is to be

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[1] It must be remembered that the expression in curly brackets in eqn. (8) represents the scalar product of two vectors, *viz.*, of the normal unit-vector and the gradient.

such that it appears anti-clockwise, when viewed from the extremity of the normal drawn from the negative to the positive side.

Eqn. (11) may also be written in the *differential* form

$$(12) \qquad d\mathbf{F} = \chi \, \mathrm{rot}_a \left(\frac{d\mathbf{s}}{r} \right),$$

the total strength of field at the field-point resulting on integration along the boundary of the double layer. Now, by § 32 eqn. (27),

$$(13) \qquad \mathrm{rot}_a \left(\frac{d\mathbf{s}}{r} \right) = \frac{1}{r} \mathrm{rot}_a \, (d\mathbf{s}) - \left[d\mathbf{s}, \mathrm{grad}_a \left(\frac{1}{r} \right) \right].$$

As the components of the curve-element are entirely independent of the coordinates of the field-point, the first term on the right-hand side of this equation obviously vanishes ; and we find, taking § 50 eqn. (4) into account,

$$(14) \qquad d\mathbf{F} = \frac{\chi}{r^2} \left[d\mathbf{s}, \frac{\mathbf{r}}{r} \right].$$

Now that we have found the rotation of the curve potential, we will proceed to calculate the *divergence of the curve potential*—again, of course, with a variable field-point. We have in the first place

$$(15) \qquad \frac{\partial P_x}{\partial x_a} = \frac{\partial}{\partial x_a} \int \frac{dx_q}{r}.$$

Since, however, the partial differentiation refers to the field-point, while the integration refers to the source, we may interchange the order of these two independent operations, thus obtaining

$$(16) \qquad \frac{\partial P_x}{\partial x_a} = \int \frac{\partial}{\partial x_a} \left(\frac{1}{r} \right) dx_q.$$

Again, by § 50 eqn. (6), the partial derivatives with respect to x_a and x_q are equal and opposite ; and hence

$$(17) \quad \mathrm{div}_a \, \mathbf{P} = - \int \left\{ \frac{\partial}{\partial x_q} \left(\frac{1}{r} \right) dx_q + \frac{\partial}{\partial y_q} \left(\frac{1}{r} \right) dy_q + \frac{\partial}{\partial z_q} \left(\frac{1}{r} \right) dz_q \right\}.$$

The expression in curly brackets is, however, simply the complete differential of $1/r$ for a fixed field-point and variable source. Therefore

$$(18) \qquad \mathrm{div}_a \, \mathbf{P} = 0,$$

provided that the curve is closed.

We will now suppose the closed curve, which we have hitherto been considering, to be expanded into a *tube* of

small, but yet *finite cross-section*. In considering the curve potential for a tube of this kind, with a cross-section q, the potential being further multiplied by a constant scalar S, we may put

(19) $$S \, d\mathbf{s} = \mathbf{A} \, d\tau \, ;$$

where $d\tau$ denotes an elementary volume of the tube, and

(20) $$\mathbf{A} = \frac{S}{q} \frac{d\mathbf{s}}{ds} \, .^2$$

The direction of the vector $\mathbf{A}$ introduced in eqn. (19) is thus that of the element of the curve, while its magnitude is equal to the quotient of the scalar S and the section of the tube.

Accordingly we have

(21) $$S\mathbf{P} = \int \frac{S \, d\mathbf{s}}{r} = \int \frac{\mathbf{A} \, d\tau}{r} \, ;$$

and, as S is constant within the tube, we see at once from § 51 eqn. (11) that

(22) $$S \, \nabla^2 \mathbf{P} = - \, 4\pi \mathbf{A} \, .$$

On the other hand, by § 32 eqn. (25),

(23) $$\text{rot rot } \mathbf{P} = \text{grad div } \mathbf{P} - \nabla^2 \mathbf{P} \, .$$

In the case of a *closed tube* we thus obtain, in accordance with eqn. (18), the important relation

(24) $$S \text{ rot rot } \mathbf{P} = 4\pi \mathbf{A} \, ,$$

the vector $\mathbf{A}$ being determined by eqn. (20) in terms of the scalar S and the section of the tube. As the divergence of a rotation always vanishes [according to § 32 eqn. (24)], it follows from eqn. (24) that, in every instance,

(25) $$\text{div } \mathbf{A} = 0 \, ,$$

provided that the tube is *closed*.

§ 55. Coulomb Distance Forces.

We shall define a *force acting between two sources* as a *Coulomb distance force*, when the force $\mathbf{K}$ exerted upon a source II, whose strength is g', by a source I is given by

(1) $$\mathbf{K} = g' \, \mathbf{F} \, ;$$

where $\mathbf{F}$ denotes the *strength of field* due to the first source at the point where the second source is situated.

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[2] For $d\tau$ is equal to $q \, ds$.

Corresponding to the force is a *potential energy* which may be called V, the force appearing as the negative gradient of V. The potential energy is thus connected with the potential Ψ of the field-strength by the relation

(2) $$V = g' \Psi$$

where Ψ is the potential in the position of the second source.

Hence we find for the *potential energy of a system of sources*

(3) $$V = \frac{1}{2} \sum_{k=1}^{k=n} g_k \Psi_k = \frac{1}{2} \sum_{h=1}^{h=n} \sum_{k=1}^{k=n} \frac{g_h \, g_k}{r_{hk}}.$$

The presence of the factor $\frac{1}{2}$ is due to the fact that each com-

bination of two sources occurs twice in the double sum (*e.g.*, $g_h \, g_k$ is equivalent to $g_4 \, g_7$, both when $h = 4$ and $k = 7$, and when $h = 7$ and $k = 4$).

If we now consider a field with a continuous distribution of sources, we have as a corresponding relation to eqn. (3)

(4) $$V = \frac{1}{2} \int \varrho \Psi \, d\tau ,$$

or, in accordance with Poisson's Equation,

(5) $$V = - \frac{1}{8\pi} \int \Psi \, \nabla^2 \Psi \, d\tau .$$

On the assumption that the field is unlimited, and that potential and strength of field both vanish at infinity (which will be the case when all the sources lie within a finite region), we then have the necessary conditions satisfied for the validity of the special *theorem of Green* [§ 33 eqn. (7)]. Consequently,

(6) $$\int \Psi \, \nabla^2 \Psi \, d\tau = - \int (\operatorname{grad} \Psi)^2 \, d\tau ,$$

or, because the negative gradient of Ψ represents the strength of field ($\mathbf{F}$),

(7) $$V = \frac{1}{8\pi} \int F^2 \, d\tau .$$

Hence the energy per unit of volume—the so-called *energy density*—is given by

(8) $$\eta = \frac{F^2}{8\pi} .$$

It follows from eqn. (2) that the potential of the mechani-

cal force acting in a field upon a *source-couple* whose strength is g' is given by

$$(9) \qquad V = g' \, (\Psi_2 - \Psi_1),$$

where Ψ_1 and Ψ_2 are the potentials at the points where the negative and positive sources are respectively situated. By § 10 eqn. (10), however,

$$(10) \qquad \Psi_2 - \Psi_1 = \mathbf{a}_{12} \, \mathrm{grad}_a \, \Psi \, ;$$

and, since the product of the connecting line $\mathbf{a}_{12}$ and g' represents the moment $\mathbf{M}'$ of the source-couple, we thus obtain

$$(11) \qquad V = \mathbf{M}' \, \mathrm{grad}_a \, \Psi \, .$$

This formula readily enables us to calculate the potential of the *mechanical force* mutually exerted by *two double-layers*. Let us consider an element of surface df' in the second double-layer, with a density of moment χ' and a unit normal $\mathbf{n}'$ (directed from the negative to the positive side). Then, by eqn. (11),

$$(12) \qquad dV = \mathbf{n}'\chi' \, \mathrm{grad}_a \, \Psi \, df' \, .$$

On the other hand, $\mathrm{grad}_a \, \Psi$ is equal and opposite to the strength of field $\mathbf{F}$, while $(-\mathbf{F})$ is equal to $(-\chi \, \mathrm{rot} \, \mathbf{P})$, by § 54 eqn. (11), where χ is the density of moment of the first double-layer, and $\mathbf{P}$ is the curve potential along its boundary. Hence

$$(13) \qquad V = - \, \chi\chi' \int \mathbf{n}' \, \mathrm{rot} \, \mathbf{P} \, df' \, .$$

Now, by *Stokes's theorem*, the integral in this equation is nothing else than the line-integral of the vector $\mathbf{P}$ taken along the boundary of the second double-layer. Denoting the element of this latter boundary by $d\mathbf{s}'$, we therefore have

$$(14) \qquad V = - \, \chi\chi' \int \mathbf{P} \, d\mathbf{s}' \, .$$

Finally, giving the curve potential of the first double-layer its value in this equation, we obtain

$$(15) \qquad V = - \, \chi\chi' \int \int \frac{d\mathbf{s} \, d\mathbf{s}'}{r} \, ,$$

where $d\mathbf{s}$ is the element of the boundary of the first double-layer.

PART II

**THEORY OF THE ELECTROMAGNETIC FIELD
AND OF LIGHT**

ELECTRICITY AND MAGNETISM

§ 56. The Electric Charge.

THE *foundations* of an *exact theory of electrical phenomena* were laid in 1785 by *Coulomb*, who was the first to establish a *quantitative law* for the phenomena of *electrical attraction and repulsion*, which had been discovered long before his time.[1] He was able to do this by means of the *torsion balance* which he invented, demonstrating that the *force* exerted by two electrically charged bodies upon each other, just as in the case of gravitation, is *inversely proportional to the square of the distance between them* and acts *in the direction of the line joining them*—provided that all other conditions remain constant.

Coulomb, moreover, discovered a second important fact bearing upon the mutual action between *different* electrical bodies at the same distance. He proved that the *force* of attraction or repulsion is *proportional* to the *product of the quantities of electricity* on the two bodies. This fact may be illustrated in the following manner. We take any *four* electrical bodies of the same kind and compare, by means of the torsion balance, the forces mutually exerted between them, *two at a time*, at *one and the same distance* r_0 apart. We shall find that, in absolute amount,

$$(1) \qquad K_{12} \times K_{34} = K_{13} \times K_{24} = K_{14} \times K_{23} ,$$

where K_{12} denotes the force acting between the first and second bodies, *etc.* It follows from eqn. (1) that, if h and i denote any two of the numbers 1, 2, 3, 4, the force K_{hi} is given by

$$(2) \qquad K_{hi} = \frac{e_h \, e_i}{r_0{}^2} ,$$

where e_h and e_i are *constants* independent of the distance, and characteristic of the *electrical state* of the two bodies numbered h and i.

It at once appears that the measurement of the forces

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[1] The power of attraction of amber, when rubbed, was known to *Thales* (about 600 B.C.): an observation of electrical repulsion was first recorded by *Guericke* (about 1650).

exerted by *three* different electrical bodies upon one another, at any arbitrary distances, is sufficient for the calculation of the quantities e_1, e_2, e_3. We only require the following equations as our basis :—

$$(3) \qquad K_{12} = \frac{e_1 e_2}{r'^2}, \qquad K_{13} = \frac{e_1 e_3}{r''^2}, \qquad K_{23} = \frac{e_2 e_3}{r'''^2}.$$

In these equations the forces and distances are experimentally determined, and so we may solve them for the three unknowns e_1, e_2, e_3. When the forces and lengths are given in absolute units, the three quantities e_1, e_2, e_3 are defined to be the *quantities of electricity*, measured in *absolute electrostatic units*, wherewith the three bodies are *charged*. The absolute unit of quantity of electricity, or the so-called *electrostatic unit*, is hence that quantity which exerts a force of one dyne upon a second quantity of equal magnitude at a distance of one centimetre. Thus the dimensions of a quantity of electricity are gm.$^{1/2}$ cm.$^{3/2}$ sec.$^{-1}$

Of fundamental importance for the theory of electricity is a discovery made by *du Fay* in 1733, *viz.*, that there are *two opposite kinds of electricity, positive* or *vitreous*, and *negative* or *resinous* [2]; and that electrical bodies of the *same kind repel* one another, while those of *opposite kind attract* one another. This important experimental fact may be expressed in the formulæ of electrical theory by ascribing a positive *sign* to positive quantities of electricity, and a negative sign to negative quantities ; and, further, in accordance with the fact that the electric force acts in the direction of the connecting line, by giving eqn. (2) a *vectorial* form. We thus obtain

$$(4) \qquad\qquad \mathbf{K} = \frac{ee'}{r^2} \frac{\mathbf{r}}{r},$$

where $\mathbf{K}$ is the force exerted by charge e upon charge e', and $\mathbf{r}$ the directed line drawn from one charge to the other. When e and e' have the same sign, it follows from eqn. (4) that the force has the same direction as $\mathbf{r}$, and is therefore one of repulsion ; but when e and e' are opposite in sign, it follows that the force is opposite in direction to $\mathbf{r}$, and is hence one of attraction, as is confirmed by experience.

The term " *quantity* of electricity " is justified by the experimental fact that the physical quantity hitherto denoted by e represents an *additive quantity*. For, if we

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[2] It was, of course, purely a matter of choice that *Franklin* (about 1750) defined vitreous electricity as positive, and resinous electricity as negative.

connect two bodies charged with quantities of electricity e_1 and e_2 in such a way that, from an electrical point of view, they may then be regarded as one single body,[3] the quantity of electricity on the latter is equal to the *sum* $(e_1 + e_2)$. Hence we can always equate the quantity of electricity on any arbitrary body to the sum of the quantities on its constituent parts : *e.g.*, the quantity of electricity in a charge continuously distributed in space is equal to the sum of the charges in the individual elements of volume. A further important experimental fact is that the *algebraical sum* of the whole of the *quantities of electricity* in a closed *system* remains *unaltered*, when the *distribution of the electricity* within the system is *varied*.[4]

§ 57. The Electrostatic Field.

If we are given an electric charge of magnitude e, we can associate with every point in its neighbourhood a vector which determines the *electric field-strength* :

$$(1) \qquad \mathbf{E} = \frac{e}{r^2} \frac{\mathbf{r}}{r},$$

where $\mathbf{r}$ is the directed line drawn from the charge to the field-point.

Let us suppose that there is a second electric charge e' at

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[3] *E.g.*, by connecting two conductors with a wire.

[4] Of course, not all physical quantities are additive quantities. Such, for example, is not the case with velocity or temperature or many other quantities. Inasmuch as *mechanical mass* affords the typical example of an additive physical quantity, it would also be possible to express the experimental fact that quantities of electricity are additive, their algebraical sum remaining constant in a closed system, by ascribing to electricity a *material character* analogous to that of ponderable matter. It might be assumed that a particular, indestructible, so-called *imponderable electric fluid* existed, whose presence in a body was the cause of an electrical condition.

As a matter of fact, similar hypotheses had already been proposed before Coulomb's discoveries, and were built up into theories, on the one hand into the *two-fluid theory* of *Symmer*, and on the other hand into the *single-fluid theory* of *Franklin*. The former theory assumed the existence of two kinds of electric fluid, one positive and the other negative, which were supposed to be mixed in bodies ; according as the one or the other kind prevailed, the body would be positively or negatively electrified, while, with equal amounts of both fluids, it would appear neutral ; electrification would thus consist in a separation of the two fluids. Franklin's theory assumed the existence of only one electric fluid ; according as a body contained it to a certain normal extent, or to a greater or lesser degree, the body would appear neutral, positively, or negatively electrified. Without taking into account the correspondence of the fundamental assumptions in these hypotheses to actual fact (or the lack of it), they both afforded comparatively useful *pictures* in describing electrostatic phenomena ; and so, even nowadays, there is a predilection for the fluid hypotheses in elementary presentations of electrostatics.

the field-point ; then the force exerted on this by the first charge is given, according to § 56 eqn. (4), by

(2) $$\mathbf{K} = e' \mathbf{E} .$$

It follows from § 55 eqn. (1) that we may at once apply the generally valid results, obtained in the theory of potential, to electric fields as well, with any arbitrary number of sources and any arbitrary distribution of the latter. In doing this we replace strength of source by electric charge, the general strength of field by the electric strength of field (**E**), and density by *density of charge* for which, as usual, the symbol ϱ will be retained. We will define the vector lines of electric field-strength as *electric lines of force*, and the flux of the vector as *flux of electric force.*

Electric strength of field and density of charge are accordingly connected by the relation [§ 51 eqn. (3)]

(3) $$\operatorname{div} \mathbf{E} = 4\pi\varrho .$$

Again, by § 50 eqn. (7), electric field-strength may be represented as the negative gradient of an *electric potential.* Retaining the symbol Ψ for the latter, we therefore have

(4) $$\mathbf{E} = - \operatorname{grad}_a \Psi ,$$

and hence, by § 32 eqn. (23),

(5) $$\operatorname{rot} \mathbf{E} = 0 .$$

The *energy density* throughout the electric field is, by § 55 eqn. (8), equal to the *square of the electric field-strength* divided by 8π ; *i.e.,*

(6) $$\eta = \frac{E^2}{8\pi} .$$

The mechanical force which acts on a unit of volume, or the *density of force* (denoted by $\mathbf{q}$), follows from eqn. (2) :

(7) $$\mathbf{q} = \varrho\mathbf{E} ,$$

or, by eqn. (3),

(8) $$\mathbf{q} = \frac{1}{4\pi} \mathbf{E} \operatorname{div} \mathbf{E} .$$

Writing this equation in its analytical instead of its vectorial form, we have

$$4\pi q_x = E_x \frac{\partial E_x}{\partial x} + E_x \frac{\partial E_y}{\partial y} + E_x \frac{\partial E_z}{\partial z} ,$$

or,

(9)
$$\begin{cases} 4\pi q_x = \frac{\partial}{\partial x} \left(\frac{E_x^2}{2} \right) + \frac{\partial}{\partial y} (E_x E_y) + \frac{\partial}{\partial z} (E_x E_z) \\ \qquad\qquad - E_y \frac{\partial E_x}{\partial y} - E_z \left(\frac{\partial E_x}{\partial z} \right) . \end{cases}$$

But, by eqn. (5),

$$(10) \qquad \frac{\partial E_x}{\partial y} = \frac{\partial E_y}{\partial x}, \quad \frac{\partial E_x}{\partial z} = \frac{\partial E_z}{\partial x};$$

so that

$$(11) \qquad \begin{cases} 4\pi q_x = \dfrac{\partial}{\partial x}\left(\dfrac{E_x{}^2}{2}\right) + \dfrac{\partial}{\partial y}(E_x E_y) + \dfrac{\partial}{\partial z}(E_x E_z) \\ \qquad\qquad - \dfrac{\partial}{\partial x}\left(\dfrac{E_y{}^2}{2}\right) - \dfrac{\partial}{\partial x}\left(\dfrac{E_z{}^2}{2}\right). \end{cases}$$

Referring back to our earlier considerations of § 36, we see that we can regard the *density of force* as the *vector divergence* of a *tensor* which we may call the tensor of *electrostatic stress*, and whose components of the first kind have the values

$$p_{xx} = \frac{1}{8\pi}(E_x{}^2 - E_y{}^2 - E_z{}^2), \text{ etc.,}$$

or, by eqn. (6),

$$(12) \quad p_{xx} = \frac{E_x{}^2}{4\pi} - \eta, \quad p_{yy} = \frac{E_y{}^2}{4\pi} - \eta, \quad p_{zz} = \frac{E_z{}^2}{4\pi} - \eta;$$

while

$$(13) \qquad\qquad p_{xy} = E_x E_y, \text{ etc.}$$

At any given point in the electric field we will now so choose the arbitrary coordinate system that its x-axis coincides with the direction of the electric field-strength at the point in question. In that case

$$(14) \qquad\qquad E_x = E, \quad E_y = E_z = 0;$$

and hence, bearing in mind eqn. (6),

$$(15) \qquad p_{xx} = \eta, \quad p_{yy} = p_{zz} = -\eta, \quad p_{xy} = 0.$$

Thus the tangential stresses vanish; while, of the principal stresses, those in the direction of x appear as tensions (on account of the positive sign), and the other two, in the directions of y and z, appear as pressures (because of the negative sign).

Hence, in an electrostatic field, there is a *tension along the lines of force*, and an equally large *lateral pressure*. Tension and pressure are *equal to the energy density* in magnitude. Following a conception introduced by *Faraday* in 1852 we may therefore regard the *lines of force* as a *dynamic system*; the tension along the lines seeks to extend them, while the transversal pressure tends to squeeze them together.

Moreover, the electric field-strength being the negative gradient of the potential, it forms an instance of a lamellar vector, and the simplest method of graphically representing an electrostatic field (according to § 34) is by the construction of the *lamellæ* formed by the *potential surfaces*. The electric field-strength is everywhere perpendicular to the potential surfaces and, in magnitude, inversely proportional to the thickness of the lamellæ. Each charge e gives rise to $4\pi e$ lines of force (further multiplied by an arbitrary proportional factor) ; or, when the charge is negative, that number of lines terminates in it.

§ 58. Distribution of Electricity on Conductors.

There is a group of bodies which, as experience shows, are peculiar in that chance *potential differences*, which may occur in them, are *immediately wiped out*. Equilibrium is thus impossible in such bodies as long as different portions thereof possess different potentials. Bodies possessing this property are known as *conductors*, and are defined by the condition that their *potential is constant throughout*, *i.e.*, that

$$(1) \qquad\qquad \mathbf{E} = 0,$$

and also

$$(2) \qquad\qquad \varrho = 0.$$

Consequently, there can be no electric charges in the *interiors* of conductors ; and, if a conductor be charged, the *electricity* must be situated wholly *on the surface of the conductor*. This important fact was first experimentally demonstrated by *Coulomb* (about 1785), the conduction of electricity having been previously discovered by *Gray* in 1729.

Denoting the surface density of electrification by σ, we find from § 52 eqn. (5) for the *potential* due to a conductor at any given field-point

$$(3) \qquad\qquad \Psi = \int \frac{\sigma \, df}{r}.$$

When the field-point lies on the surface of the conductor the integral in eqn. (3) gives the constant value of the potential in the conductor.

The *potential energy* of a conductor follows from § 55 eqn. (3) :—

$$(4) \qquad\qquad V = \frac{1}{2} \int \sigma \Psi \, df.$$

But the value of the potential must be *the same* over the whole surface of the conductor, while the integral

$$(5) \qquad \int \sigma \, df = Q$$

represents the total electric charge on it. Hence

$$(6) \qquad V = \frac{1}{2} \Psi Q .$$

If, now, *equilibrium* exists with a definite distribution of electricity over the surface of the conductor, it must also exist when the density of electrification is increased or diminished throughout in the same proportion. Comparing eqns. (3) and (5) we see that there must be *proportionality between charge and potential*, the proportional factor depending only on the size and shape of the conductor, while it is completely independent of the electrical condition of the latter. This factor, which, in accordance with eqns. (3) and (5), must have the *dimensions of a length*, is known as the *capacity* of the conductor. Denoting it by C, we thus have

$$(7) \qquad C = \frac{Q}{\Psi},$$

and hence, by eqn. (6),

$$(8) \qquad V = \frac{1}{2} \frac{Q^2}{C} .$$

We may easily arrive at the magnitude and direction of the *strength of field* at the surface of a conductor by means of two relations previously obtained (§ 52). According to these, the tangential components of the field-strength are identical in value on either side of the surface, while the normal component, in the direction of the outwards-drawn normal, is $4\pi\sigma$ units greater on the outside than on the inside. Since the strength of field vanishes throughout the interior of a conductor, the tangential component on the outside must also vanish. As, however, the outwards-directed normal component is $4\pi\sigma$, we find at the surface of a conductor

$$(9) \qquad \mathbf{E} = 4\pi\sigma\mathbf{n} ,$$

where $\mathbf{n}$ is the outwards-drawn normal unit-vector.

As a very simple *example*, we shall now consider the case of a *spherical conductor*. The *potential* on the surface of such a conductor is given by eqn. (3) as

$$\Psi = \int \frac{\sigma \, df}{r} :$$

i.e., we take any given point P on the surface, and then r denotes the distance of this point P from the individual elements of the surface. On account of the complete symmetry of the sphere, the integral $\int \dfrac{df}{r}$ obviously has always the same value, wherever the point P be chosen on the surface. Moreover, if the potential is to have the same value for every arbitrary point on the spherical surface, the surface
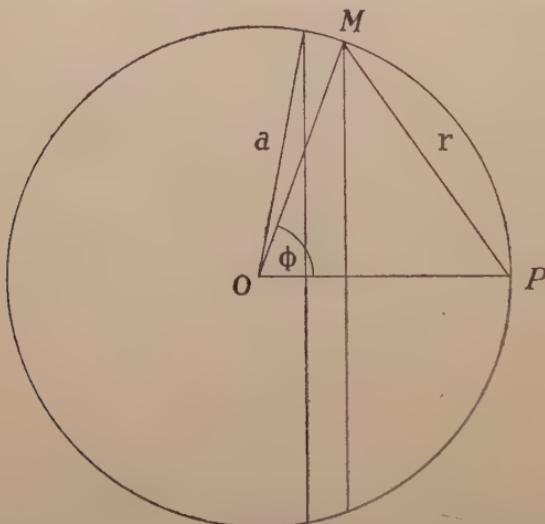

Fig. 46.

density must also be the same throughout. Consequently, we may place the quantity σ before the symbol of integration, in calculating the potential.

The locus of all points on the surface of the sphere, whose distances from P lie between r and $(r + dr)$, is, as Fig. 46 shows, a spherical belt with radii $a \sin \varphi$ and $a \sin (\varphi + d\varphi)$, and of width $a\, d\varphi$, where a is the radius of the sphere. The area of this belt is therefore

(10) $$df = 2\pi a \sin \varphi \,.\, a\, d\varphi\,.$$

As a consideration of the triangle POM shows us, the value of φ may be obtained from the relation

$$r^2 = 2a^2 - 2a^2 \cos \varphi\,.$$

We have

$$r\, dr = a^2 \sin \varphi\, d\varphi\,,$$

and therefore
$$df = 2\pi r\, dr\,.$$
Hence
$$(11) \qquad \Psi = \sigma \int_0^{2a} \frac{2\pi r\, dr}{r} = 4\pi a\sigma\,.$$

On the other hand, the charge on the spherical conductor is given by

$$(12) \qquad Q = 4\pi a^2 \sigma\,.$$

Substituting the values given by eqns. (11) and (12) in eqn. (7) we find the simple relations

$$(13) \qquad C = a$$

and

$$(14) \qquad \Psi = Q/a\,.$$

The *capacity* of a spherical conductor is simply *equal to its radius*.[1]

Let us now consider an uncharged conductor situated in an electrostatic field. If we construct two congruent surfaces I and II (Fig. 47) parallel and very close to any portion of the surface bounding the conductor, there will be a flux of electric force, $\int E_n\, df$ in amount, through surface I lying in the air, where **E** is the value of the strength of field at the surface of the conductor. On the other hand, there will be *no flux of force* through surface II lying within the conductor, seeing that the electric field-

Fig. 47.

strength is zero throughout its interior. We may thus say that the *flux of electric force terminates* on the outside of the conductor. But we may also interpret the facts by supposing the portion of the surface under consideration to be itself charged with a quantity of electricity such that the outward flux of force, due to this quantity, is equal and opposite to that entering the portion in question.[2] It follows

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[1] An entirely analogous consideration shows us that the potential, due to a spherical conductor, at any arbitrary point P' outside the sphere is equal to the quotient of the total charge and the distance OP'. An electrically charged spherical conductor thus behaves, with respect to a point lying outside it, as if the whole charge were concentrated at the centre of the sphere.

[2] For the whole of the flux of force proceeding from the surface of a conductor must be in an outward direction.
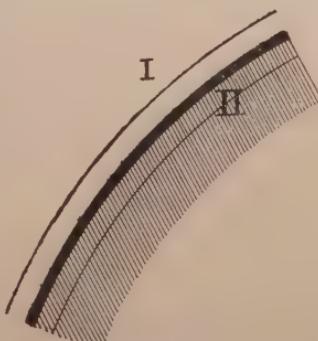
from § 50 eqn. (14) that the amount of this so-called *induced electricity* is determined by the relation

$$(15) \qquad 4\pi e' = -\int E_n \, df \,.$$

We will next consider a *spherical* conductor with an (actual) electric charge e. A flux of force of total amount $4\pi e$ proceeds uniformly in all directions from it. Let us now suppose this conductor to be enclosed within a second, *concentric*, conducting spherical surface of radius b. It follows from what we have just said that a charge of total amount $(-e)$ will appear induced upon the inner side of this second surface. If the latter be insulated, and if it possesses no proper electric charge of its own, then, by § 56, the algebraical sum of all the electricity contained in it cannot differ from zero ; consequently, the outer side of the larger surface becomes charged with induced electricity of the same sign as the original charge, and of amount $(+e)$. If, however, the larger sphere be *earthed*, only the induced electricity of opposite sign remains on its inner surface.

An apparatus consisting of two concentric spherical conducting shells, of which the outer is connected to earth while the inner can be electrically charged, is known as a *spherical condenser*. If the inner sphere be charged with a quantity of electricity Q, we have as potential of the spherical condenser, from eqn. (14),

$$(16) \qquad \Psi = \frac{Q}{a} - \frac{Q}{b} = Q\frac{b-a}{ab} \,.$$

The capacity of a spherical condenser is therefore given by

$$(17) \qquad C = \frac{ab}{b-a} \,,$$

so that the capacity of the condenser is unconditionally greater than that of the inner sphere. By enclosing the inner sphere within the outer sphere and connecting the latter to earth, we enable the inner sphere to take up a greater electric charge than would be possible, at the same potential, without this outer envelope.[3] As is usually stated, electricity is " condensed " by this enclosing shell.[4]

[3] If we connect the inner sphere with the conductor of an influence machine, electricity will flow over to the former until it has the same potential as the conductor.

[4] The conditions in the so-called plate condenser are exactly similar.

§ 59.   The Electric Current.

Electricity in motion implies a *flow of electricity*. In particular we speak of a *current in a conductor*, or simply of an *electric current*, when the movement takes place within a *cylindrical* conductor, *e.g.*, within a *wire*. The rate of flow of electricity past any section of such a wire is defined as the *current strength*, and may be denoted by I. In accordance with § 54 eqn. (20), the current may be associated with a vector whose magnitude is equal to the quotient of the current and the cross-section of the wire, and whose direction is that of the wire. This vector quantity is known as the *current density* and is denoted by $\mathbf{i}$; we thus have

$$(1) \qquad\qquad \mathbf{i} = \frac{I}{q}\frac{d\mathbf{s}}{ds},$$

where $d\mathbf{s}$ is a curve-element of the wire. The direction of the current, however, will coincide with that of the flow of electricity, or be directly opposed to it, according as the electricity in motion is positive or negative.

It follows from the definition of current density that, in the case of *closed currents* [in accordance with § 54 eqn. (25)], which alone we shall be considering in what follows, we have

$$(2) \qquad\qquad \operatorname{div}\mathbf{i} = 0.$$

Now we may regard the wire in which the current is flowing as the vector tube of the vector $\mathbf{i}$ (in the sense of § 34), seeing that no electricity passes through its walls. Since, by eqn. (2), $\mathbf{i}$ is a solenoidal vector, while the product of its magnitude and the cross-section of the wire gives the current, it follows from § 34 eqn. (3) that the *value of the current* is *everywhere the same* throughout a *closed circuit*.

Gauss's theorem enables us at once to deduce from eqn. (2) another important law, which holds for the case of *branched currents*. If we suppose a closed surface to be constructed about the junction of the branches, the integral of i_n taken over this surface must vanish, as follows from Gauss's theorem and eqn. (2). This integral, however, simply represents the algebraical sum of all the currents. Hence the *sum of the currents in the branches* must be *equal to the main current*. This relation was first discovered by *Kirchhoff* in 1845.

Let us now consider a length of a current-bearing con-

ductor bounded by two sections A and B, the current flowing from A to B. We define the quantity

$$(3) \qquad \Psi_A - \Psi_B = \int_A^B \mathbf{E}\, d\mathbf{s}$$

as the *electromotive force* acting over the length AB. In accordance with the definition of potential, the potential difference in eqn. (3) at the same time represents the *work* performed when an electrostatic unit is moved from A to B.

The electromotive force of a current is defined as the line-integral of the electric field-strength around the circuit, the sense of integration being that of the direction of the current. The integral is then always positive, for the current always flows in the direction of the electric field-strength, so that the scalar product $\mathbf{E}\, d\mathbf{s}$ is also invariably positive.

The electromotive force of the current is equal to the work performed by an electrostatic unit in flowing once round the circuit. Since, in one unit of time, every section of the conductor is traversed by I electrostatic units, where I is the current, it follows that the *work performed by the current in a unit of time,* or the so-called *power,* is given by

$$(4) \qquad w = I \int \mathbf{E}\, d\mathbf{s}.$$

§ 60. Magnetism.

Let us consider the *vector field* formed by the *rotation of the curve potential* of a current (I) flowing in a *closed circuit.* In accordance with § 54 eqn. (11) this field is, apart from a proportional factor, identical with that of the *strength of field* produced by a *double-layer encircled by the current.*

We now come to the *second fundamental law of the theory of electricity*, alongside the law of electrostatic attraction, and it may be expressed in the following words :—*Two closed electric currents* exert on *each other* a *mechanical force* which is *equal to the Coulomb distance force between two double-layers encircled by the currents*, provided that the *density of moment* of each double-layer is made *equal to the quotient of the current and a universal constant*, which proves to be a *velocity of* 3×10^{10} cm./sec.

Denoting this universal constant by c, we can thus replace every closed plane circuit by a double-layer whose total *moment* is

$$(1) \qquad \mathbf{M} = \mathbf{n}\frac{I}{c}f,$$

where f is the area of the surface enclosed by the current, or the *circuit area*.[1] According to § 54, the sense of the normal unit-vector must be such that, when viewed from its extremity, the sense in which the curve potential of the current is formed appears anti-clockwise. But, in forming the curve potential, the integration is carried out in the sense of the current, and hence the direction of the latter must appear anti-clockwise when viewed from the extremity of the vector **n**.

Making use of a long-established expression we shall now define the equivalent double-layer by which every electric current may be in imagination replaced, as regards its effect on other currents, as a *magnetic shell*. Correspondingly, we speak of the *magnetic moment of a current*, as given by eqn. (1). If we suppose the shell to have a definite arbitrary thickness (a), we can define the quotient of the magnitude of the moment **M** and the thickness a as the *quantity of magnetism* which is present in equal amount on each of the two opposite faces. Joining the two faces by a normal vector whose sense is that of the vector **n**, the current flowing in an anti-clockwise direction when viewed from the latter's extremity, we must (in accordance with § 53) define as *negatively magnetic* that face from which the vector starts, and as *positively magnetic* that towards which it is directed.

Theoretical physics may thus assume, as a counterpart to electricity, a *fictitious agency* known as *magnetism*, which, just like electricity, is of a *dual nature* and exerts Coulomb distance forces between its various parts (in the sense of § 55). The analogy between electricity and magnetism is, however, limited by the fact that only magnetic source-*couples* appear to be possible, and these always in the nature of *surfaces*. But none the less we can also apply the results of our earlier general considerations to magnetism, provided that we now substitute quantity of magnetism for strength of source, and *magnetic strength of field* (denoted by **H**) for strength of field in general.

We shall call the universal constant c the *electromagnetic constant*. It follows immediately from eqn. (1) that it must have the *dimensions of a velocity*. For, inasmuch as quantities of magnetism and of electricity have the same dimensions, the dimensions of the left-hand side of eqn. (1) are quantity of electricity × length. On the right-hand side **n** has zero dimensions, while the current I is, by

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[1] The general case of a curved current-surface can be reduced to the special case of a plane surface, in the sense of Fig. 30.

definition, the quotient of a quantity of electricity and a time. Hence the electromagnetic constant actually has the dimensions of a velocity. The quotient I/c may be termed the current in electromagnetic measure, while I itself will always represent the quotient of a quantity of electricity and a time.

Let us now picture to ourselves a body containing within itself an extremely large number of very minute currents ; and let us consider an elementary volume of the body $d\tau$ which, however, still contains a very large number of these elementary currents. We may then form the vectorial sum of the moments of all the elementary currents contained in this volume-element ; and, on dividing the vector thus obtained by the magnitude of the element of volume, we arrive at a vector which may be called the *vector of specific magnetization* (denoted by the symbol h). When this vector has the same value throughout the interior of the body, we say that the body is *homogeneously magnetized*.

We will now proceed to calculate the *magnetic potential* due to a homogeneously magnetized body at any given field-point. According to § 53 eqn. (3), the potential of an elementary volume is

$$(2) \qquad d\Psi = \mathbf{h} \operatorname{grad}_q \left(\frac{1}{r} \right) d\tau .$$

Again, by § 32 eqn. (26),

$$(3) \qquad \mathbf{h} \operatorname{grad}_q \left(\frac{1}{r} \right) = \operatorname{div}_q \left(\frac{\mathbf{h}}{r} \right) - \frac{1}{r} \operatorname{div}_q \mathbf{h} .$$

As, however, h is constant for a homogeneously magnetized body, the last term in eqn. (3) vanishes ; and we thus find, on integrating eqn. (2),

$$(4) \qquad \Psi = \int \operatorname{div}_q \left(\frac{\mathbf{h}}{r} \right) d\tau ,$$

or, by *Gauss's theorem*,

$$(5) \qquad \Psi = \int \frac{h_n}{r} df ,$$

where df is an element of the surface of the homogeneously magnetized body, and the normal component is taken with reference to the outwards-drawn normal to the surface.

Comparing the above equation with § 52 eqn. (5), we see that a *homogeneously magnetized body* behaves as though the *magnetism were located solely on the surface*, with the surface

density everywhere equal to the normal component of the
vector of specific magnetization. If, in particular, we con-
sider a cylindrical bar whose axis coincides with the direc-
tion of this vector, the normal component (and consequently
the surface density) vanishes all along its sides. Moreover,
the direction of the outwards-drawn normal is opposite in
sense at the two end surfaces, as it has the direction of the
vector **h**. Thus a cylindrical bar of this kind, which is known
as a *bar magnet*, exhibits *magnetism only at its two ends*.
The two end surfaces are called the two *poles* of the magnet,
and the quantity of magnetism present on each of them is
termed the *pole strength*. The pole strength is equal to the
product of the area of the end surface and the magnitude of
the vector of specific magnetization (intensity of magne-
tization), taken with a positive or negative sign as the case
may be.

A spiral conductor with a rectilinear axis which is
traversed by a current—a so-called *solenoid*—will obviously
exhibit the same properties as a bar magnet, for to each
turn of the spiral there corresponds a moment given by
eqn. (1). Let Z be the total number of turns; then the total
moment of the solenoid is given by

$$(6) \qquad \mathbf{M} = \mathbf{n}\, Z \frac{I}{c} f.$$

On dividing the magnitude of the vector **M** by the length of
the spiral, we obtain the quantity of magnetism m which is
located on either end surface. But the quotient of the
number Z and the length of the spiral gives the number of
turns per unit of length; hence, denoting the latter number
by z, we have

$$(7) \qquad m = \pm\, z \frac{I}{c} f.$$

In accordance with a previous definition, the positive pole
will be situated at that end from which the current appears
to flow through the spiral in an anti-clockwise direction.

If we now suspend a solenoid of this kind so that it is
free to move in a horizontal plane, it will take up such a
position, under the influence of the earth's magnetism, that
its positive pole points towards the North. Hence, distin-
guishing in traditional fashion the two poles of a magnet as
north pole and south pole (according as the pole points to
the North or the South), we see that the previously defined
conception of *positive* magnetism coincides with the con-
ventional conception of *north-polar magnetism*.

§ 61. The Law of Biot and Savart.

The *magnetic strength of field* produced by a *closed electric current* at any given *field-point* follows at once from § 54 eqn. (11), the density of moment (χ) being now replaced by the electromagnetically measured current (I/c) (in accordance with § 60). We thus find for the magnetic strength of field

$$(1) \qquad \mathbf{H} = \frac{I}{c} \operatorname{rot}_a \int \frac{d\mathbf{s}}{r},$$

where the integral signifies the curve potential of the current.

According to § 54 eqn. (14), we may also write this equation in the *differential form*

$$(2) \qquad d\mathbf{H} = \frac{I}{cr^2} \left[d\mathbf{s}, \frac{\mathbf{r}}{r} \right],$$

$d\mathbf{H}$ thus denoting the field-strength due to an individual *element of current* $d\mathbf{s}$. Hence this strength of field is, in magnitude,

$$(3) \qquad dH = \frac{I}{c} \frac{ds \sin \theta}{r^2},$$

where θ is the angle included between the line joining the

FIG. 48.

current-element to the field-point and the direction of the element (Fig. 48). The law expressed by eqn. (3) was deduced in 1820 by *Biot* and *Savart* from experimental results, which referred, of course, not to elementary currents, but only to complete currents. This theoretical deduction followed closely u p o n *Oersted's* discovery, in the same year, of the deviation of a magnetic needle by an electric current.
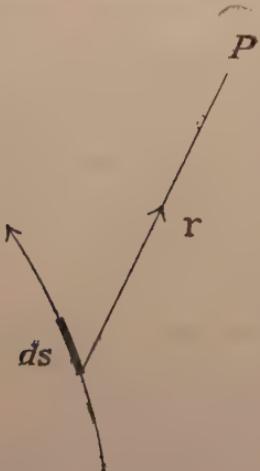
The *direction* of the vector $d\mathbf{H}$ is determined by the fact that it is *perpendicular to the plane* containing the curve-element and the field-point. If we suppose this plane to be rotated, together with the field-point, about the curve-element, we see at once that the *magnetic lines of force* due to a *rectilinear* portion of a current-bearing conductor must be *circles* whose centres lie in the conductor.

Finally, the *sense* of $d\mathbf{H}$ is determined by the definition of a vector product and by eqn. (2) to be such that, when viewed from its extremity, the rotation necessary to carry the curve-element ds round into the direction of the vector $\mathbf{r}$ appears anti-clockwise. Thus, in Fig. 48, the vector $d\mathbf{H}$ will be directed backwards. If we imagine a human figure to be swimming in the current, with face turned towards the field-point, then the outstretched *left arm* will give the direction of the vector $d\mathbf{H}$. This so-called *swimmer's rule* for determining the sense of the magnetic deviation was suggested by *Ampère*, also in 1820.

It follows from eqn. (3) that the magnetic strength of field created at its centre by a *circular current* of radius a is given by

$$(4) \qquad H = \frac{I}{c}\frac{2\pi}{a},$$

for $\sin \theta$ is here equal to 1. Since pole strengths can be determined from measurements of the forces of attraction or repulsion, on the basis of Coulomb's Law, eqn. (4) affords a simple means of determining *currents in electromagnetic measure* (using a so-called galvanometer), and thus of evaluating the quotient I/c.[1] In 1856 *Weber* and *Kohlrausch* succeeded in finding both the quantities I/c and I in the case of the discharge current of a Leyden jar ; this was done by measuring the charge of the jar before discharge, as well as the time-integral of the discharge current.[2] They could thus calculate the *electromagnetic constant c*, and found a value of $3 \cdot 1 \times 10^{10}$ cm./sec. ; later measurements led to the more exact value of $3 \cdot 0 \times 10^{10}$ cm./sec.

As the divergence of a rotation must always vanish, there follows from eqn. (1) the very important relation

$$(5) \qquad \operatorname{div} \mathbf{H} = 0 .$$

On the other hand, we have [by § 54 eqn. (24) and § 59 eqn. (1)] the following relation for the curve potential of a closed current :—

$$(6) \qquad I \operatorname{rot} \operatorname{rot} \mathbf{P} = 4\pi\mathbf{i} .$$

[1] The electromagnetic unit of current is thus a current which, flowing in a circular conductor one unit of length in radius, exerts a force of 2π dynes on a unit quantity of magnetism placed at the centre of the circle. The unit quantity of magnetism is defined as being such that a unit magnetic pole attracts or repels a second equally strong pole, at a distance of one centimetre, with a force of one dyne.

[2] The charge was measured by means of an electrometer ; and the time integral of the discharge current (which is equal to the charge) was measured with a so-called ballistic galvanometer.

Again, by eqn. (1),

$$(7) \qquad \frac{I}{c} \operatorname{rot} \mathbf{P} = \mathbf{H} ;$$

and so, on dividing eqn. (6) throughout by c, we obtain the important formula

$$(8) \qquad \operatorname{rot} \mathbf{H} = \frac{4\pi}{c} \mathbf{i} ,$$

which is generally known as the *first principal equation of the electromagnetic field*.

This equation may also be expressed in another form, the current density being replaced by the current itself. For, if we consider a *curve* which goes *once completely round a current conductor* (Fig. 49), it follows from *Stokes's theorem* that the *line-integral of the magnetic strength of field* along this curve is given by
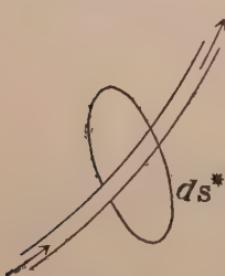
$$(9) \qquad \int \mathbf{H} \, d\mathbf{s}^* = \int \mathbf{n}^* \operatorname{rot} \mathbf{H} \, df^* .$$

Fig. 49.

In this expression $d\mathbf{s}^*$ denotes an element of the curve encircling the conductor, df^* an element of the surface enclosed by this curve, and $\mathbf{n}^*$ the unit-normal to this surface; in accordance with Stokes's theorem the sense of the unit-normal will be such that, when viewed from its extremity, the sense of integration of the line-integral appears anti-clockwise. Inserting on the right-hand side of eqn. (9) the value for the rotation of the magnetic field-strength given in eqn. (8), we find

$$(10) \qquad \int \mathbf{H} \, d\mathbf{s}^* = \frac{4\pi}{c} \int i_n \, df^* .$$

But the surface-integral of the normal component of the current density is simply the quantity of electricity passing the surface in one unit of time; and, seeing that the electricity flows only in the conductor, this quantity is the current I (taken with a positive sign when the sense of integration of the line-integral appears anti-clockwise, as seen from the extremity of the vector $\mathbf{i}$). Hence

$$(11) \qquad \int \mathbf{H} \, d\mathbf{s}^* = \frac{4\pi}{c} I .$$

Thus the *work* done when a *unit pole travels once round a*

circuit is *equal to the current*, electromagnetically measured, multiplied by 4π.

We may also apply this relation in the very simple calculation of the *magnetic field-strength in the interior of a long solenoid*. Let us consider a closed curve which, within the solenoid, runs parallel to its axis, and which passes through the end surfaces of the solenoid. When the latter is very long we may regard the strength of field within it as constant ; and we may further, by way of approximation, equate the line-integral of the strength of field along the closed curve to the product of this constant field-strength and the length (a) of the axis of the solenoid. If Z be the total number of turns, then, by eqn. (11), seeing that each turn is encircled once by the curve,

$$(12) \qquad\qquad Ha = 4\pi Z \, \frac{I}{c}.$$

Denoting, further, the number of turns per unit of length by z, we find within the solenoid

$$(13) \qquad\qquad H = 4\pi z \, \frac{I}{c},$$

whereby the sense of the strength of field (whose direction is parallel to the axis) is determined by Ampère's swimmer's rule.

§ 62. Ampère's Fundamental Law of Electrodynamics.

The *potential of the mechanical force* mutually exerted between *two electric currents* follows at once from § 55 eqn. (15), if (in accordance with § 60) the density of moment (strength of shell) be replaced by the current, measured electromagnetically. We find in this way the relation established by *Ampère* :

$$(1) \qquad\qquad V = -\frac{II'}{c^2} \int\int \frac{d\mathbf{s}\,d\mathbf{s}'}{r}.$$

This equation may be written in another form, by making use of § 55 eqn. (13) and § 61 eqn. (7) :—

$$(2) \qquad\qquad V = -\frac{I'}{c} \int \mathbf{n}'\mathbf{H}\,df'.$$

The *potential energy of a current in a magnetic field* is thus equal and opposite to the *product of the current* (in electro-magnetic measure) and the *flux of magnetic force through the surface enclosed by the circuit.*

By considering the mutual interaction of two elements of two currents we can write eqn. (1) in a *differential form*. We shall investigate the special case in which the two current-elements are *parallel*, their senses being either the same or opposed. The scalar product in eqn. (1) then simply becomes equal to the arithmetical product with a positive sign, when the two elements are parallel and have the same sense. We therefore have

$$(3) \qquad dV = -\frac{II'}{c^2}\frac{ds\,ds'}{r}.$$

Now let $d\mathbf{K}$ be the force exerted by the first element (ds) upon the second (ds'), and let $\mathbf{r}$ be the directed line drawn from the former to the latter. As dV is a function only of r, for given currents, we find for the mechanical force as negative gradient of the potential [in accordance with § 10 eqn. (18)]

$$(4) \qquad d\mathbf{K} = -\frac{II'}{c^2}\frac{ds\,ds'}{r^2}\frac{\mathbf{r}}{r}.$$

Hence, when the two current-elements are similarly directed, the force exerted by the first element on the second is opposite in sense to the directed line drawn from the former to the latter. *Similarly directed current-elements* thus *attract each other*, whereas *oppositely directed ones repel each other*. This important fact was experimentally discovered in 1820 by *Ampère*, who also showed that the force of attraction or repulsion varies directly as the product of the currents and the lengths of the elements, and inversely as the square of the distance between them—all of which is expressed by eqn. (4).

Fig. 50.

Eqn. (2) enables us also readily to calculate the *mechanical force* exerted *upon a movable portion of a current conductor* placed *in a magnetic field*. We shall assume an element of the conductor $(d\mathbf{s})$ to be displaced in such a way that the area of the circuit is increased, the displacement being represented by the vector $d\mathbf{a}$ (Fig. 50) ; in this process, however, the current shall not be interrupted. (In this imaginary experiment we might, for instance, suppose the element of the conductor to roll over conducting rails without friction.)

The displacement of the conductor-element will then
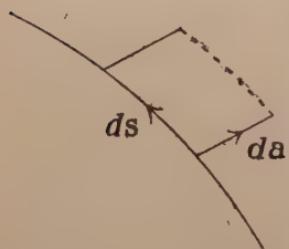
produce an increment in the area of the circuit of magnitude

(5) $df = ds\, da \sin (d\mathbf{a}, d\mathbf{s})$.

Let **n** be the unit-vector normal to this surface-element, its sense being such that, when viewed from its extremity, the direction of the current appears anti-clockwise ; thus, in Fig. 50, the vector **n** will point forwards from the plane of the paper. But the vector product $[d\mathbf{a}, d\mathbf{s}]$, whose magnitude is equal to df, will have the same direction and sense, so that

(6) $\mathbf{n}\, df = [d\mathbf{a}, d\mathbf{s}]$.

If the conductor-element be situated in a magnetic field whose strength is **H**, it follows from eqn. (2) that the increase in area of the circuit will be accompanied by a change of potential energy in amount given by

(7) $dV = -\dfrac{I}{c}\, \mathbf{n}\mathbf{H}\, df$,

or,

(8) $dV = -\dfrac{I}{c}\, \mathbf{H}\, [d\mathbf{a}, d\mathbf{s}]$.

In accordance with a formula of vectorial algebra [§ 3 eqn. (23)] which we have repeatedly used, this may also be written

(9) $dV = -\dfrac{I}{c}\, d\mathbf{a}\, [d\mathbf{s}, \mathbf{H}]$.

Again, denoting the mechanical force by **K**, we have the elementary mechanical relation [§ 11 eqn. (4)]

(10) $dV = -\, \mathbf{K}\, d\mathbf{a}$.

Since the equality of the right-hand sides of eqns. (9) and (10) must hold for *any direction* of the arbitrary vector $d\mathbf{a}$, we thus obtain, quite generally, the relation

(11) $\mathbf{K} = \dfrac{I}{c}\, [d\mathbf{s}, \mathbf{H}]$.

This equation determines the mechanical force exerted upon an element of a conductor conveying a current in a magnetic field ; this force, which is proportional to the current, acts therefore at right angles both to the direction of the element and to the magnetic lines of force.

In accordance with § 59 eqn. (1) we may write

(12) $I\, d\mathbf{s} = \mathbf{i}\, d\tau$,

where **i** is the current density and $d\tau$ a volume-element.

Denoting the *density of force in the magnetic field* by $\mathfrak{q}$, we have, by eqn. (11),

$$(13) \qquad \mathfrak{q} = \frac{1}{c}\,[\mathbf{iH}]\,,$$

or by § 61 eqn. (8),

$$(14) \qquad 4\pi\mathfrak{q} = [\text{rot}\ \mathbf{H},\ \mathbf{H}]\,.$$

Now

$$(15) \quad \begin{cases} [\text{rot}\ \mathbf{H},\ \mathbf{H}]_x = H_z\,\text{rot}_y\,\mathbf{H} - H_y\,\text{rot}_z\,\mathbf{H} \\[2mm] \qquad = H_z\dfrac{\partial H_x}{\partial z} - H_z\dfrac{\partial H_z}{\partial x} - H_y\dfrac{\partial H_y}{\partial x} + H_y\dfrac{\partial H_x}{\partial y} \\[3mm] \qquad = -\dfrac{\partial}{\partial x}\left(\dfrac{H_y{}^2 + H_z{}^2}{2}\right) + \dfrac{\partial}{\partial y}(H_x\,H_y) \\[3mm] \qquad + \dfrac{\partial}{\partial z}(H_x\,H_z) - H_x\left(\dfrac{\partial H_y}{\partial y} + \dfrac{\partial H_z}{\partial z}\right). \end{cases}$$

But, by § 61 eqn. (5),

$$(16) \qquad \frac{\partial H_y}{\partial y} + \frac{\partial H_z}{\partial z} = -\frac{\partial H_x}{\partial x}\,,$$

so that

$$(17) \quad \begin{cases} 4\pi q_x = \dfrac{\partial}{\partial x}\left(\dfrac{H_x{}^2}{2}\right) + \dfrac{\partial}{\partial y}(H_x\,H_y) + \dfrac{\partial}{\partial z}(H_x\,H_z) \\[3mm] \qquad\qquad - \dfrac{\partial}{\partial x}\left(\dfrac{H_y{}^2 + H_z{}^2}{2}\right). \end{cases}$$

This equation, however, agrees completely with the quite analogous equation derived for the density of force in an electrostatic field [§ 57 eqn. (11)], except that the electric strength of field is here replaced by the magnetic field-strength.

Hence there exists a *tension along the magnetic lines of force,* and an *equally large lateral pressure,* whereby both tension and pressure are equal in magnitude to the *square of the magnetic strength of field, divided by* 8π.

§ 63. Neumann's Law of Induction.

When the *magnetic flux of force* passing through a circuit *varies,* § 62 eqn. (2) indicates that there must always be a simultaneous *change in potential energy,* which, as *experience* shows, finds its *equivalent* in the appearance of an *induced current* in the conductor. The quantitative laws bearing on these experimental facts may be very simply derived from the Principle of the *Conservation of Energy.* According to this principle, the work done by the induced current in an

element of time dt must obviously be equal to the change in potential energy during the same interval. Provided that we regard the *induced current* as *steady*, *i.e.*, constant in strength, we find for the change in potential energy [according to § 62 eqn. (2), and omitting the dash in the quantities **n** and df as redundant in this section]

$$(1) \qquad dV = -\frac{I}{c} dt \; \frac{d}{dt} \int \mathbf{n} \mathbf{H} \, df.$$

In this expression the sense of the normal unit-vector **n** must be such that, when viewed from its extremity, the direction of the induced current appears anti-clockwise.

On the other hand, by § 59 eqn. (4),

$$(2) \qquad dV = I \, dt \int \mathbf{E} \, d\mathbf{s}.$$

Comparing eqns. (1) and (2) we obtain the important relation

$$(3) \qquad \int \mathbf{E} \, d\mathbf{s} = -\frac{1}{c} \frac{d}{dt} \int \mathbf{n} \mathbf{H} \, df.$$

The *induced electromotive force* is therefore, apart from a proportional factor, *equal to the time-rate of change of the magnetic flux of force through the area of the circuit*. This so-called *law of induction* was established in 1845 by *Franz Neumann*, subsequently to the discovery by *Faraday* in 1831 of the phenomenon of induced currents. As eqn. (3) shows, the more rapid the change of the magnetic flux of force (or, in other words, of the number of magnetic lines of force passing through the circuit), the stronger is the induced current.

The *sense of the induced current* also follows from eqn. (3). In § 59 we defined electromotive force as the line-integral of the strength of field, taken in the sense of the latter ; now since the line-integral is thus always positive, the time-derivative of the integral on the right-hand side of eqn. (3) must always be negative. The sense of the normal unit-vector **n**, from whose extremity the induced current will appear to flow in an anti-clockwise direction, must therefore always be such that the magnetic *flux of force decreases* relatively to it.

If, for example, we bring the positive pole of a magnet towards a closed conductor, the magnetic flux of force relative to the normal **n** can only decrease when the normal unit-vector is directed towards the approaching positive pole. so that it forms an *obtuse angle* with the lines of force

arising from the latter (Fig. 51). When, however, the normal **n** has such a sense, the positive side of the magnetic shell (§ 60) equivalent to the induced current will also be facing the approaching pole. Hence, in accordance with Coulomb's Law, the *approach of the pole is resisted* by a force of repulsion.

Any given case whatsoever may be reduced to the special case just considered, whether the induced current be due to the approach or the withdrawal of a magnet or a current, to the strengthening or the weakening of a magnet or a current, or to the opening or the closing of a primary current. In eqn. (3) we thus recognize a law of general validity, which was empirically established by *Lenz* in 1834, and which is commonly called, after him, Lenz's Law. This states that the sense of the induced current is always such that the magnetic force arising from it tends to *check the inducing process.*

FIG. 51.

Eqn. (3) assumes an especially simple form in the special case that the *surface enclosed by the current is rigid and at rest.* In eqn. (1) we may then replace the total time-derivative by a partial one ; and we may further interchange the order of the two mutually independent operations of partial differentiation with respect to the time and integration over the surface. We thus have

$$(4) \qquad \frac{d}{dt} \int \mathbf{n} \mathbf{H} \, df = \int \mathbf{n} \frac{\partial \mathbf{H}}{\partial t} \, df .$$

Again, by *Stokes's theorem,*

$$(5) \qquad \int \mathbf{E} \, d\mathbf{s} = \int \mathbf{n} \, \mathrm{rot} \, \mathbf{E} \, df .$$

Inserting in eqn. (3) the values given by eqns. (4) and (5), we obtain an equation which must be satisfied for any arbitrary shape and position of the circuit : thence results the important relation

$$(6) \qquad \mathrm{rot} \, \mathbf{E} = - \frac{1}{c} \frac{\partial \mathbf{H}}{\partial t} .$$

This relation is known as the *second principal equation of the electromagnetic field.*
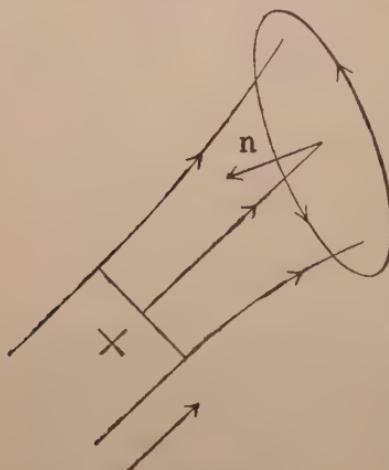
§ 64. Ohm's Law.

The two principal equations of the electromagnetic field can both be derived without the necessity of utilizing the knowledge of the relation connecting current with electromotive force. Nevertheless, the question of this relation is of fundamental importance in all special investigations on electric conduction currents ; and, as a matter of fact, an empirical solution was reached by *Ohm* as long ago as 1827.

We may express Ohm's Law by stating that the *vector of the current density* in a conductor is *proportional to the electric field-strength*, the proportional factor (which is called the *specific conductivity*) depending only on the nature of the conductor.[1] Thus, denoting the specific conductivity by λ, we have

$$(1) \qquad \mathbf{i} = \lambda \mathbf{E}.$$

Multiplying this equation by $q\,ds$, where q is the cross-section of the conductor and ds is the length of an element of it, we find by § 59 eqn. (1)

$$(2) \qquad I\,d\mathbf{s} = \lambda q \mathbf{E}\,ds.$$

Multiplying again by the unit-vector $d\mathbf{s}/ds$, we obtain [2]

$$(3) \qquad I\,ds = \lambda q \mathbf{E}\,d\mathbf{s}.$$

Let us now integrate this equation over a length (l) of the conductor ; then

$$\int ds = l,$$

and

$$\int \mathbf{E}\,d\mathbf{s} = K,$$

where K is the electromotive force acting between the limits of the conductor. We thus find

$$(4) \qquad I = \frac{\lambda q}{l} K.$$

Ohm's Law may hence be expressed by the usual formula

$$(5) \qquad K = RI,$$

[1] The value of λ, for one and the same material, depends in general on the temperature. In the case of selenium, the value, as is well known, varies considerably under exposure to light ; in the case of bismuth it changes when the conductor is brought into a magnetic field. Moreover, Ohm's Law is in no sense to be regarded as an axiom of electrical theory, but merely as an empirically obtained expression of a relation, for the theoretical derivation of which we must revert to assumptions from the electron theory of metals.

[2] For $(d\mathbf{s})^2 = (ds)^2$.

where the quantity R, which is termed the *resistance*, is given by the relation

$$(6) \qquad R = \frac{1}{\lambda} \frac{l}{q}.$$

Electromotive force is therefore *equal to the product of current and resistance*, while, in its turn, *resistance varies directly as the length* of the conductor and *inversely* as its *cross-section*.

Ohm's Law leads at once to the relation known as *Kirchhoff's second law of branched currents*. If a current divides up anywhere into branches which reunite further on, these branches are said to be "connected in parallel"; moreover, since the electromotive force is the same for all branches, the currents in these must be inversely proportional to the respective resistances.

It follows from Ohm's Law that the *work done by a current in a unit of time* is, in accordance with § 59 eqn. (4),

$$(7) \qquad W = I^2 R.$$

When no mechanical work is done by the current, this amount of energy is dissipated as *heat*. The amount of heat developed in a unit of time is thus proportional to the resistance and the square of the current, as was discovered in 1841 by *Joule*.[3]

Joule's Law may be expressed in another way by substituting for the current, on the one hand, the product of the magnitude of the current density and the cross-section of the conductor, and for the resistance, on the other hand, its value as given in eqn. (6). We then have

$$(8) \qquad W = q^2 i^2 \frac{l}{q\lambda} = lq \frac{i^2}{\lambda}.$$

The reciprocal value of the specific conductivity is known as the specific resistance. The rate at which heat is developed *in a unit of volume* is therefore determined by the product of the specific resistance and the square of the current density.

In experimental physics currents are generally measured in "*ampères*," potentials and electromotive forces in "*volts*." These *practical units* are quite arbitrarily chosen. A current is defined as having a strength of one ampère when 3×10^9 electrostatic units flow across a section of the conductor in a second. A volt is equal to one three-hundredth part of the absolute unit of electric potential, and this, in its turn, represents the quotient of one erg and one electrostatic unit.

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[3] The proportional factor depends only on the units employed.

The dimensions of the electrostatic unit of quantity are (by § 56) gm.$^{1/2}$ cm.$^{3/2}$ sec.$^{-1}$; dividing this by a second, we obtain the absolute unit of current.   Hence

(9)   1 ampère = 3×10^9 gm.$^{1/2}$ cm.$^{3/2}$ sec.$^{-2}$.

The dimensions of an erg are gm. cm.^{2} sec.$^{-2}$, so that

(10)   1 volt = $\dfrac{1}{300}$ gm.$^{1/2}$ cm.$^{1/2}$ sec.$^{-1}$.

The practical unit of *resistance* is the *ohm*, which is the resistance of a conductor in which an electromotive force of one volt produces a current of one ampère.   Therefore, by eqns. (9) and (10),

(11)   1 ohm = $\dfrac{1 \text{ volt}}{1 \text{ ampère}}$ = $\dfrac{1}{9 \times 10^{11}}$ cm^{-1}. sec .

Of the other practical units in electrical science only the most important will be mentioned.   The practical unit of *quantity of electricity* is the *coulomb*, which is the quantity flowing across a section of the conductor in one second when the current is one ampère.   Hence, by eqn. (9),

(12)   1 coulomb = 3×10^9 electrostatic units.

The unit of *capacity* is the *farad*, which is the capacity of a conductor that is raised to a potential of one volt by a charge of one coulomb.   Consequently,

(13)   1 farad = $\dfrac{1 \text{ coulomb}}{1 \text{ volt}}$ = 9×10^{11} cm .

The unit of *power* (or the rate of doing work) is the *watt*, which is the power developed by a current of one ampère under an electromotive force of one volt.   Therefore, by eqns. (9) and (10),

(14)   1 watt = 1 volt $\times$ 1 ampère = 10^7 erg sec.$^{-1}$.

§ 65. Self-Induction.

As a result of the mechanical force exerted upon each other by the individual elements of a conductor conveying *one and the same current*, every electric current is associated with a *potential energy* [according to § 62 eqn. (1)] of magnitude

(1) $V = -\dfrac{I^2L}{2}$,

where, for short,

$$(2) \qquad L = \frac{1}{c^2} \int\int \frac{d\mathbf{s}\,d\mathbf{s}'}{r}.$$

The elements $d\mathbf{s}$ and $d\mathbf{s}'$ are now two different current-elements belonging to the same conductor; the factor $1/2$ is necessary in eqn. (1) because each element occurs twice in the double integral, once as $d\mathbf{s}$ and once as $d\mathbf{s}'$.

Every variation in the current will thus produce an induction effect in the conductor, and this effect is known as *self-induction*. The quantity L, as defined by eqn. (2), is called the *coefficient of self-induction, or the self-inductance*; it represents an invariable quantity when the form of the conductor remains unchanged.

In accordance with § 63 eqn. (2) every variation in the potential energy is accompanied by an *induced electromotive force* [1] whose magnitude is

$$(3) \qquad K' = \frac{1}{I}\frac{dV}{dt}.$$

By eqn. (1), however,

$$(4) \qquad \frac{dV}{dt} = -\,IL\frac{dI}{dt};$$

and hence

$$(5) \qquad K' = -\,L\frac{dI}{dt}.$$

If K be the electromotive force originally present, and R the resistance, *Ohm's Law* thus assumes the form

$$(6) \qquad K = RI + L\frac{dI}{dt},$$

when the self-induction is taken into account.

In discussing this equation we will first consider the special case in which K is constant. We then find, on differentiating eqn. (6) with respect to the time,

$$(7) \qquad R\frac{dI}{dt} + L\frac{d^2I}{dt^2} = 0.$$

Regarding dI/dt as an unknown function of the time we have, as a solution of eqn. (7),

$$(8) \qquad \frac{dI}{dt} = A'e^{\alpha t},$$

[1] The practical unit of inductance is known as the *henry*. It is the value of the self-inductance of a circuit, for which a variation of the current by 1 ampère in a second induces an electromotive force of 1 volt.

where A' and α are constants. The constant α can at once be determined, for on differentiating eqn. (8) with respect to the time, we find

(9)
$$\frac{d^2I}{dt^2} = \alpha \frac{dI}{dt} \ ;$$

and, inserting this in eqn. (7), we obtain

(10) $R + L\alpha = 0 \,,$

or,

(11) $$\alpha = -\frac{R}{L} \,.$$

Again, on integrating eqn. (8) and putting, for short, A'/α equal to A,

(12) $I = Ae^{\alpha t} + B \,,$

where B is another integration constant. We may at once determine B by substituting in eqn. (6) the values given in eqns. (12) and (8) ; then[2]

(13) $K = RB + Ae^{\alpha t} (R + L\alpha) \,.$

The expression in brackets vanishes, however, in virtue of eqn. (10), so that

(14) $$B = \frac{K}{R} \,,$$

and hence eqn. (12) becomes

(15) $$I = \frac{K}{R} + Ae^{-\frac{R}{L}t} \,.$$

This equation, in which the constant A is still undetermined, holds good in the case of a *constant electromotive force* ; we shall now apply it to the special case of the *making* or *breaking of an electric circuit*. If the current be made at the time $t = 0$, we have for $t = 0$ also $I = 0$, and eqn. (15) gives us a relation to determine A :—

(16) $$0 = \frac{K}{R} + A \,.$$

Hence, by eqn. (15), in the special case under consideration,

(17) $$I = \frac{K}{R} \left(1 - e^{-\frac{R}{L}t} \right) .$$

The current gradually increases, finally reaching the limiting value K/R as determined by Ohm's Law. The growth of the

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[2] Bearing in mind that $A' = aA.$

current takes place more slowly the smaller the resistance and the greater the inductance, as is obvious from eqn. (17).

If, on the other hand, the *circuit be broken*, we must put $K = 0$. It follows from eqn. (15) that, in this instance, the constant $A = I_0$, where I_0 is the value of the current at the time $t = 0$. Hence eqn. (15) now becomes

$$(18) \qquad I = I_0 e^{-\frac{R}{L}t}.$$

The induced current due to self-induction that occurs on making or breaking a circuit is called the *extra current*.

We may make a further important application of eqn. (6), when the *electromotive force varies periodically*. Eqn. (6) then becomes

$$(19) \qquad K_0 \sin pt = RI + L \frac{dI}{dt},$$

where K_0 denotes the amplitude of the periodically varying electromotive force, while p is equal to the quotient of 2π and the period. The considerations of § 41, which are entirely analogous, suggest as a trial solution of the differential equation (19)

$$(20) \qquad I = A \sin (pt + \delta),$$

where A, the amplitude of the current, and δ, the difference in phase, are constants which we have to determine. We then have

$$(21) \qquad \frac{dI}{dt} = Ap \cos (pt + \delta).$$

On developing the sine and cosine of the angular sum in eqns. (20) and (21) according to the well-known trigonometrical formulæ, and then substituting these values in eqn. (19), we find

$$K_0 \sin pt = RA (\sin pt \cos \delta + \cos pt \sin \delta)$$
$$+ LAp (\cos pt \cos \delta - \sin pt \sin \delta),$$

or,

$$(22) \quad \left\{ \begin{array}{l} \sin pt (K_0 - RA \cos \delta + LAp \sin \delta) \\ \qquad - \cos pt (RA \sin \delta + LAp \cos \delta) = 0. \end{array} \right.$$

This equation, however, must be satisfied for *any* arbitrary value of the time; and this is only possible when *both* the expressions in brackets vanish. The vanishing of the second expression gives us

$$(23) \qquad \tan \delta = -\frac{Lp}{R};$$

while it follows from the vanishing of the first expression in brackets that

$$A (R \cos \delta - Lp \sin \delta) = K_0,$$

or,

$$A \cos \delta (R - Lp \tan \delta) = K_0.$$

Dividing by R, and taking into account eqn. (23), we thus obtain

$$A \cos \delta (1 + \tan^2 \delta) = \frac{K_0}{R};$$

and therefore [3]

$$(24) \qquad A = \frac{K_0}{R \sqrt{1 + \tan^2 \delta}},$$

or, by eqn. (23),

$$(25) \qquad A = \frac{K_0}{\sqrt{R^2 + L^2 p^2}}.$$

If we introduce the *frequency* of the electromotive force by means of the relation [4]

$$(26) \qquad p = 2\pi\nu,$$

eqns. (23) and (25) become

$$(27) \qquad \tan \delta = - \frac{2\pi\nu L}{R}$$

and

$$(28) \qquad A = \frac{K_0}{\sqrt{R^2 + 4\pi^2 \nu^2 L^2}}.$$

Inserting the values given by eqns. (23) and (25) in eqn. (20), we thus obtain an actual solution of the differential eqn. (19).

When the electromotive force in a circuit varies periodically, the current, which [by eqn. (20)] also varies periodically in strength and direction, is known as an *alternating current*. The quantity ν is called the *frequency* of the alternating current. The expression under the square root in eqn. (28), which corresponds to the resistance in the case of a direct current, is termed the *impedance* of the alternating circuit: we see from eqn. (28) that the greater the frequency and the

[3] According to the well-known trigonometrical formula

$$\cos \delta = \frac{1}{\sqrt{1 + \tan^2 \delta}}.$$

[4] *Cf.* § 41 eqn. (21).

inductance, the more the impedance exceeds the resistance.[5]
The current *lags behind* the electromotive force *in phase*;
and it follows from eqn. (27) that the greater the frequency
and the self-induction, the greater is the lag.

As a final example of the application of eqn. (6) we will
consider the *discharge of a condenser*. The potential of the
condenser must now be substituted for the electromotive
force, and so [by § 58 eqn. (7)]

$$(29) \qquad K = \frac{Q}{C},$$

where Q is the charge, and C the *capacity* of the condenser.
On the other hand, the strength of the *discharge current*
follows from the relation

$$(30) \qquad I = -\frac{dQ}{dt}.$$

Hence, for the discharge current of a condenser, eqn. (6)
becomes

$$(31) \qquad \frac{Q}{C} = -R\frac{dQ}{dt} - L\frac{d^2Q}{dt^2},$$

or,

$$(32) \qquad \frac{d^2Q}{dt^2} + \frac{R}{L}\frac{dQ}{dt} + \frac{Q}{LC} = 0;$$

or again, on differentiating with respect to the time, bearing
in mind eqn. (30) and changing all the signs,

$$(33) \qquad \frac{d^2I}{dt^2} + \frac{R}{L}\frac{dI}{dt} + \frac{I}{LC} = 0.$$

Both eqn. (32) and eqn. (33) represent *differential
equations of a damped oscillation*. Comparing them with
§ 40 eqn. (5), we see that the constant of damping

$$(34) \qquad k = \frac{R}{L},$$

while for the period of oscillation we find, by § 40 eqn. (16),

$$(35) \qquad \tau = \frac{2\pi}{\sqrt{\dfrac{1}{LC} - \dfrac{R^2}{4L^2}}}.$$

---

[5] The quantity $2\pi\nu L$ is called the *reactance*. The square of the impedance
is thus equal to the sum of the squares of the resistance and the reactance,
so that a simple graphical representation is possible.

When the resistance R is very small, eqn. (35) takes the simple form

$$(36) \qquad \tau = 2\pi\sqrt{LC}.$$

Thus both the condenser charge and the discharge current represent *oscillatory quantities*. It follows from eqn. (36) that the smaller the capacity and the inductance of the system, the more rapid are the *discharge oscillations*. Again, by eqn. (34), the smaller the resistance and the greater the inductance, the less is the damping.

The theory of condenser oscillations was first derived, in 1853, by *William Thomson* (*Lord Kelvin*) from the law of induction; the oscillatory character of the discharge was experimentally verified by *Feddersen* in 1859.

MAXWELL'S THEORY OF THE ELECTRO-MAGNETIC FIELD

§ 66. Displacement Currents.

WHILE the relations of electrostatics follow from *Coulomb's law of electric attraction*, previous considerations have shown us that both Biot and Savart's law and Neumann's law of induction may be deduced by purely mathematical means, with the help of the Principle of Energy, from the *law of magnetic attraction between closed currents*. To these two fundamental laws a *third* was added in 1873, when *Maxwell* formulated his *law of displacement currents*, which may be expressed in the following form :—

Every variation in the electric strength of field is accompanied by a displacement current in an electromagnetic field, this current possessing the same properties as a conduction current, and, either by itself or in conjunction with a conduction current, representing a *closed circuit*.

Now the connection between the rate of change of the electric field-strength at any given point, and the density of the displacement current at that point, follows from the relation which states that the *electric flux of force* through a closed surface (which we suppose rigid and stationary) is *equal to 4π times the total quantity of electricity enclosed by the surface*. We have, by § 50 eqn. (16),

$$(1) \qquad \int \mathbf{n}\mathbf{E}\, df = 4\pi \sum e \,.$$

Since we imagine the surface to be rigid and stationary, we can interchange the order of the partial differentiation with respect to the time and the integration over the surface, as being independent operations, and we thus find

$$(2) \qquad \int \mathbf{n}\, \frac{\partial \mathbf{E}}{\partial t}\, df = 4\pi \frac{\partial \left(\sum e \right)}{\partial t}\,.$$

The partial time-derivative of $\sum e$, however, simply repre-

sents the total quantity of electricity flowing in through the rigid surface in a unit of time. But, since this current is to be taken as closed, the displacement current, which flows out through the surface in the same time, must be equal and opposite in amount. Hence, denoting the density of this hypothetical displacement current by **g**, we have

$$(3) \qquad \int \mathbf{n} \frac{\partial \mathbf{E}}{\partial t} \, df = 4\pi \int \mathbf{ng} \, df.$$

This relation, however, must be satisfied quite generally for any arbitrary size and shape of the rigid surface, and we thus obtain the connection between displacement current and strength of field in the form

$$(4) \qquad \mathbf{g} = \frac{1}{4\pi} \frac{\partial \mathbf{E}}{\partial t}.$$

It was a daring conception on the part of Maxwell to ascribe to the hypothetical displacement currents exactly the *same properties* as are observed in conduction currents, but his boldness has been brilliantly justified by the extraordinary fruitfulness of the results to which it has led. A *displacement current* will therefore also *produce a magnetic field*.

Thus the *first principal equation of the electromagnetic field* [§ 61 eqn. (8)] is modified by the hypothesis of displacement currents to the relation

$$(5) \qquad \mathrm{rot} \, \mathbf{H} = \frac{4\pi}{c} (\mathbf{i} + \mathbf{g}),$$

where **i** is the density of the conduction current and **H** the magnetic strength of field. In free space, where there are no conduction currents, eqn. (5) assumes the form, when eqn. (4) is taken into account,

$$(6) \qquad \mathrm{rot} \, \mathbf{H} = \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t}.$$

§ 67. Maxwell's Equations.

The hypothesis of displacement currents forms the basis of the great *theory*, created by *Maxwell* in 1873, which was destined, through its unitary comprehension of the phenomena of electricity, of magnetism, and of light, to lead to a development of physics along entirely new paths.

Maxwell's theory is founded on *four* equations known as *Maxwell's equations of the electromagnetic field*. Of these, the first two refer to the *divergences of the electric and the magnetic strengths of field*. Maxwell's theory *transfers* the fundamental

relation of a constant electrostatic field [§ 57 eqn. (3)], according to which the divergence of the electric field-strength is equal to the density of charge multiplied by 4π, to electromagnetic fields *varying in any arbitrarily rapid manner*. In the second place it assumes, as having general validity, the relation deduced in the older theory of closed conduction currents, according to which the divergence of the magnetic field-strength always vanishes [§ 61 eqn. (5)].

The third and fourth of Maxwell's equations refer to the *rotations of the electric and the magnetic strengths of field*. The third is identical with the so-called *second principal equation* of the electromagnetic field, which has already been discussed [§ 63 eqn. (6)]. It is true that this equation was only deduced on the basis of two suppositions, first, that the *induced current* is *steady*, and secondly, that the surface enclosed by the circuit is rigid and at rest. Maxwell's theory, however, *extends* the scope of the equation to cover *arbitrarily variable currents*, and regards the second hypothesis as being satisfied when the electromagnetic field is considered as a continuously distributed region in space. Finally, the fourth equation is identical with the so-called *first principal equation* of the electromagnetic field in its extended form, already derived in the previous section from the hypothesis of displacement currents [§ 66 eqns. (5) and (6)]. Although this first principal equation only holds good in the case of closed currents, Maxwell's theory is justified in regarding it as generally valid, because, in accordance with that theory, an apparently open circuit is always completed to a closed circuit by the displacement current.

Maxwell's equations thus assume the form

$$(1) \quad \begin{cases} \operatorname{div} \mathbf{E} = 4\pi\varrho , \\ \operatorname{div} \mathbf{H} = 0 , \\ \operatorname{rot} \mathbf{E} = -\dfrac{1}{c}\dfrac{\partial \mathbf{H}}{\partial t} , \\ \operatorname{rot} \mathbf{H} = \dfrac{1}{c}\dfrac{\partial \mathbf{E}}{\partial t} . \end{cases}$$

The fourth equation is here given the simplified form which holds in the absence of conduction currents ; otherwise it becomes, when Ohm's Law [§ 64 eqn. (1)] is taken into account,

$$(2) \quad \operatorname{rot} \mathbf{H} = \frac{4\pi\lambda}{c}\mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} ,$$

where λ denotes the specific conductivity.

§ 68. Poynting's Theorem.

It follows from § 55 eqn. (8) that the *energy density* in an electromagnetic field is given by

$$(1) \qquad \eta = \frac{1}{8\pi}(E^2 + H^2),$$

where E and H are the magnitudes of the electric and the magnetic strengths of field respectively. It is a fundamental conception in *Maxwell's theory* that *every elementary volume* $(d\tau)$ is the carrier of a quantity of energy $\eta \, d\tau$, whereby, in accordance with eqn. (1), the value of η is completely determined by the values of the electric and the magnetic field-strengths at the point in question.[1]

We arrive at an important theorem by forming the scalar product of the fourth of Maxwell's equations [§ 67 eqn. (1)] and **E**, and of the third equation and **H**. Subtracting the latter from the former, we find

$$(2) \qquad \mathbf{E} \operatorname{rot} \mathbf{H} - \mathbf{H} \operatorname{rot} \mathbf{E} = \frac{1}{c} \frac{\partial}{\partial t} \left(\frac{E^2 + H^2}{2} \right).$$

Introducing eqn. (1) and § 28 eqn. (32) we have

$$(3) \qquad - \operatorname{div} [\mathbf{EH}] = \frac{4\pi}{c} \frac{\partial \eta}{\partial t}.$$

Now, just as the Principle of the Conservation of Mass was given a vectorial analytical expression in § 46 eqn. (10), so we may express the *Principle of the Conservation of Energy* in the form

$$(4) \qquad \operatorname{div}(\eta \mathbf{v}) + \frac{\partial \eta}{\partial t} = 0,$$

where **v** is the velocity of the flow of energy. Hence, introducing a vector

$$(5) \qquad \mathbf{S} = \frac{c}{4\pi} [\mathbf{EH}],$$

we see by comparing eqns. (3) and (4) that this vector represents the *density of the flow of energy*, in magnitude, in direction, and in sense.

The vector **S** is known as *Poynting's vector*, after the investigator who first (1884) introduced it into the theory.

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[1] The older theory had merely assumed that the value of the *total energy* of the field was obtained by integration of the density of energy over the *whole* field; η was thus only a mathematical quantity without real physical significance.

Eqn. (3) is an expression of *Poynting's Theorem* which states that, at every point in an electromagnetic field, there is a *flow of energy perpendicular to the directions of the electric and the magnetic strengths of field.*

§ 69. Electromagnetic Waves.

Let us consider a space free from charges; then, in consequence of this, the divergence not only of the magnetic, but also of the electric strength of field vanishes throughout. We then have, by § 32 eqn. (25),

(1) $$\begin{cases} \nabla^2 \mathbf{E} = -\operatorname{rot}\operatorname{rot} \mathbf{E}, \\ \nabla^2 \mathbf{H} = -\operatorname{rot}\operatorname{rot} \mathbf{H}. \end{cases}$$

Now, since the mutually independent operations of the formation of a rotation and of partial differentiation with respect to time may have their order interchanged, it follows from § 67 eqn. (1) that

(2) $$\begin{cases} \operatorname{rot}\operatorname{rot} \mathbf{E} = -\dfrac{1}{c}\dfrac{\partial}{\partial t}(\operatorname{rot} \mathbf{H}) = -\dfrac{1}{c^2}\dfrac{\partial^2 \mathbf{E}}{\partial t^2}, \\ \operatorname{rot}\operatorname{rot} \mathbf{H} = \dfrac{1}{c}\dfrac{\partial}{\partial t}(\operatorname{rot} \mathbf{E}) = -\dfrac{1}{c^2}\dfrac{\partial^2 \mathbf{H}}{\partial t^2}. \end{cases}$$

Introducing these values into eqn. (1) we thus obtain

(3) $$\nabla^2 \mathbf{E} = \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2}, \qquad \nabla^2 \mathbf{H} = \frac{1}{c^2}\frac{\partial^2 \mathbf{H}}{\partial t^2}.$$

In accordance with § 43, these two equations at once indicate the possibility of a *propagation of the electric and the magnetic field-strengths in plane waves,* whose *velocity* in free space must be *equal to the electromagnetic constant.* As the divergence of the magnetic strength of field always vanishes, the magnetic waves (as they may be briefly termed) at any rate must be *purely transversal,* as follows from § 43. The same will hold for the electric waves in such regions as contain no free electric charges, and where, in consequence, the divergence of the electric strength of field vanishes throughout.

Let us now consider a plane transverse electric wave, whose direction of propagation is along the x-axis; then, in the most general case of elliptical vibrations, the electric wave may be described by the equations

(4) $$\begin{cases} E_y = A \sin [a\,(x - ct) + \delta], \\ E_z = B \sin [a\,(x - ct)\,]. \end{cases}$$

Here A, B, a, δ are constants ; A and B are the amplitudes of the two linear and mutually perpendicular partial vibrations into which the elliptical vibrations may be resolved with respect to the coordinate axes ; δ is the difference in phase between the partial vibrations ; and a is equal to 2π divided by the wave-length [cf. § 42 eqn. (10)].

Since the partial derivatives of the components of **E** with respect to y and z vanish in the plane wave, Maxwell's third equation [§ 67 eqn. (1)] may be written analytically as follows :—

(5) $\dfrac{\partial H_x}{\partial t} = 0 , \qquad \dfrac{1}{c}\dfrac{\partial H_y}{\partial t} = \dfrac{\partial E_z}{\partial x} , \qquad \dfrac{1}{c}\dfrac{\partial H_z}{\partial t} = - \dfrac{\partial E_y}{\partial x} .$

Giving E_y and E_z the values obtained in eqn. (4), we have

(6)
$$\begin{cases} \dfrac{1}{c}\dfrac{\partial H_y}{\partial t} = Ba \cos\left[a\left(x - ct\right)\right], \\[2mm] \dfrac{1}{c}\dfrac{\partial H_z}{\partial t} = - Aa \cos\left[a\left(x - ct\right) + \delta\right]. \end{cases}$$

Integration of these equations leads to the relations

(7)
$$\begin{cases} H_y = - E_z + \Phi_1, \\ H_z = E_y + \Phi_2, \end{cases}$$

where Φ_1 and Φ_2 may be functions of the coordinates, but must be independent of the time. We can, however, omit these functions if we assume that no magnetic forces were present at the time $t = 0$. Hence, multiplying the first of the two eqns. (7) by E_y and the second by E_z, and then adding, we find

(8) $E_y H_y + E_z H_z = 0 .$

On introducing into eqns. (7) the values given for E_z and E_y in eqn. (4) (while Φ_1 and Φ_2 are simply put equal to zero), we see that *with every electric wave is associated a magnetic wave of the same period and with the same direction of propagation.* Now it follows from the first of the three eqns. (5) that the x-component of the magnetic strength of field does not partake in the vibrations ; hence, if we assume that there were no magnetic forces originally present in the region under consideration, H_x must always be zero. Thus, on adding the vanishing product $E_x H_x$ to the left-hand side of eqn. (8), the latter becomes (in vectorial form)

(9)   **EH** $= 0 :$

i.e., the *vectors of the electric and the magnetic strengths of field* are *perpendicular to each other in an electromagnetic plane wave.*

When, in particular, the direction of the vector **E** is that of the positive y-axis (permanently in the case of a linear vibration, and at a certain moment in the case of an elliptical vibration), E_z will be zero, and E_y positive. Hence, by eqn. (7), H_y will vanish, while H_z will be positive. The senses of the propagation of the waves, of the electric and of the magnetic field-strengths therefore bear the same relation to one another as the senses of the positive x-, y-, and z-axes in an *English coordinate system*—and this is the reason why, in modern physics, the English coordinate system is given preference to the French. The rotation
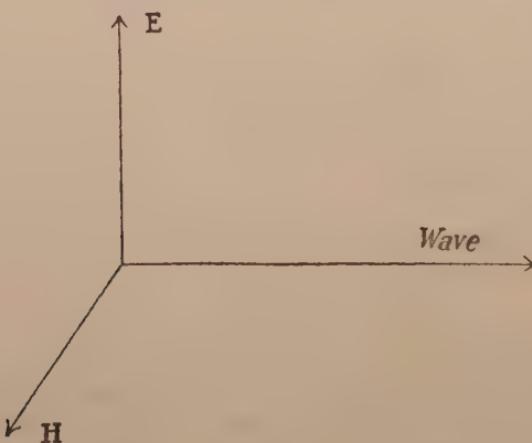
FIG. 52.

necessary to carry the vector **E** round into the direction of the vector **H** thus appears anti-clockwise, when viewed from the extremity of a vector whose direction and sense is that of the propagation of the waves (Fig. 52). Hence, by § 68 eqn. (5), the *direction of Poynting's vector* coincides with that of the *waves*.

Since the direction of propagation of the waves is perpendicular both to the electric and to the magnetic lines of force, it follows from the previous considerations of §§ 57 and 62 that a *pressure* must exist in the direction of propagation which will be equal to the sum of the squares of the magnetic and the electric strengths of field, divided by 8π. This pressure, which is termed the *electromagnetic radiation pressure*, is therefore *equal to the density of electromagnetic energy* [as determined by § 68 eqn. (1)].

Thus, if we consider a fixed surface which absorbs the

waves and is perpendicular to their direction of propagation, an element (df) of this surface will be acted on by a force whose direction is that of the waves, and whose magnitude is given by

$$(10) \qquad\qquad K = \eta \, df.$$

Now, instead of an electromagnetic wave, let us take the case of a fluid whose density is ϱ^* and whose velocity of flow is v. In one unit of time a quantity of liquid $\varrho^* v \, df$ will flow against the elementary surface, and the momentum imparted to the latter in that time will thus be $\varrho^* v^2 \, df$. In accordance with Newton's Second Law of Motion, this rate of increase in momentum will be equal to the force acting on the surface-element. Hence, denoting this force by K',

$$(11) \qquad\qquad K' = \varrho^* v^2 \, df.$$

Comparing eqns. (10) and (11), we see that we can *ascribe inertial mass to the electromagnetic energy of an electromagnetic wave.* As the energy is propagated with the velocity of the wave, *i.e.*, with the velocity c, we find the mass density of the energy to be

$$(12) \qquad\qquad \varrho^* = \frac{\eta}{c^2}.$$

The inertial *mass* of the energy of electromagnetic waves is therefore obtained by *dividing the energy by the square of the electromagnetic constant.*

§ 70. Convection Currents.

Since by *Maxwell's theory only closed currents* can exist, it was a necessary consequence of the development of this theory that *every moving body*, if it be *electrically charged*, must produce a *magnetic field* exactly as does a conduction current, and conversely, it will be subjected to a mechanical force when placed in a magnetic field. This conclusion was *experimentally* verified in 1876 as a result of an important research carried out by *Rowland*. A disc fitted with charged sectors of tinfoil was set in rapid rotation, and Rowland found that a magnetic needle hung in its proximity was deflected.

About 1880, Maxwell's theory was given a most important extension by *Lorentz*, who put forward the hypothesis that every electrically charged body in motion represents a so-called *convection current*. The hypothesis of convection currents established, to a certain extent, the idea of an *electromechanical parallelism*. One and the same process,

the motion of an electrically charged mass could now be considered from two different aspects. It was now possible to apply the laws both of *mechanics* and of the *electromagnetic field* to this one process ; and it is just to this *simultaneous applicability of the laws of two different branches of physical science* that a whole series of new and fruitful relations is due.

Let ϱ be the density of electric charge and $\mathbf{v}$ the mechanical velocity ; then the *density of the convection current* is given by

$$(1) \qquad \mathbf{i} = \varrho \mathbf{v} \,.$$

It follows from § 62 eqn. (13) that the density of force in a magnetic field is

$$(2) \qquad \mathbf{q} = \frac{\varrho}{c} \, [\mathbf{vH}] \,.$$

Hence we find for the *mechanical force* exerted by a *magnetic field* upon a body charged with a quantity of electricity Q, and moving with a velocity $\mathbf{v}$,

$$(3) \qquad \mathbf{K}_1 = \frac{Q}{c} \, [\mathbf{vH}] \,.$$

Moreover, if the body be situated in an electromagnetic field, it is also acted on by an *electrostatic force* given by

$$(4) \qquad \mathbf{K}_2 = Q\mathbf{E} \,,$$

where $\mathbf{E}$ is the electric field-strength.

§ 71. The Equation of Motion of an Electrically Charged Body.

With the results of the previous section as a basis, we will now investigate more fully the *motion of an electrically charged body in an electromagnetic field*. In doing so we shall disregard gravitational forces, by assuming that the *mass of the body* is so *small* in comparison with its charge that *gravitation* may be *neglected* alongside the electric and magnetic forces.

We will define the *quotient of the electric charge of the body and its mass* as the *specific charge* and denote it by γ. It follows from *Newton's Second Law of Motion* and from § 70 eqns. (3) and (4) that the *equation of motion* of the electrically charged body takes the form

$$(1) \qquad \frac{d\mathbf{v}}{dt} = \gamma \mathbf{E} + \frac{\gamma}{c} \, [\mathbf{vH}] \,.$$

We will now discuss this equation in *two special cases* ; first, for the case of a purely electric field, and secondly, for that of a purely magnetic field : in the first case **H**, and in the second **E**, vanishes.

For a *purely electric field* eqn. (1) becomes

$$(2) \qquad \frac{d\mathbf{v}}{dt} = \gamma \mathbf{E} .$$

If the *electric field be constant and homogeneous*, the *acceleration* given by the left-hand side of eqn. (2) will also be *constant* both in magnitude and direction. Hence, in a constant and homogeneous electric field, the charged body will move in exactly the same way as an inertial body in a gravitational field. If it has an initial velocity, the body will thus describe a parabola.[1]

On specializing still further, and assuming that the original direction of motion is perpendicular to the direction of the electric strength of field—we then speak of a *transverse* electric field—we find that a complete *analogy* exists with *horizontal projection*. Let that point on the path where the body is situated at the time $t = 0$ be chosen as the origin, and let the direction of motion at the time $t = 0$ be the x-axis, while the direction of the strength of field is the z-axis ; then, by § 5 eqn. (6), the equation of the parabola is given by [2]

$$(3) \qquad z = \frac{\gamma E}{2} \frac{x^2}{u^2} ,$$

where u is the velocity at the time $t = 0$.

Our second special case is that of a *purely magnetic field*. Eqn. (1) then becomes

$$(4) \qquad \frac{d\mathbf{v}}{dt} = \frac{\gamma}{c} [\mathbf{vH}] .$$

The *vector of acceleration* is therefore always *perpendicular to the direction of motion* as determined by the vector **v**. Hence, resolving the acceleration (according to § 4) into a tangential component (b_t) and a normal component (b_n), we have

$$(5) \qquad\qquad b_t = 0$$

and

$$(6) \qquad\qquad b_n = b ,$$

---

[1] Naturally the path will only be a parabola when the direction of the initial velocity does not coincide with that of the electric strength of field ; otherwise it will be a straight line, as in the case of vertical projection.

[2] We replace $(-g)$ in § 5 eqn. (6) by γE, whilst, in the special case of horizontal projection, the angle of elevation vanishes.

where b, as usual, denotes the magnitude of the acceleration. Since, by § 4 eqn. (11), b_t is equal to the time-derivative of the magnitude of the velocity, we find that

$$(7) \qquad\qquad v = \text{const.}$$

During *motion in a magnetic field* the *velocity remains constant in magnitude*, so that there is *no change in kinetic energy* and hence *no work is done*.

For the normal component of the acceleration we have, by eqns. (4) and (6),

$$(8) \qquad\qquad b_n = \frac{\gamma}{c}\, vH \sin (\mathbf{v}, \mathbf{H}) ;$$

while, by § 4 eqn. (13),

$$(9) \qquad\qquad b_n = \frac{v^2}{r} ,$$

where r is the radius of curvature of the path.

We will now make a further condition, and assume the direction of the magnetic field-strength to be perpendicular to the original direction of motion of the body on entering the field. We then speak of a *transverse magnetic field*. If $\mathbf{v}$ is originally at right angles to $\mathbf{H}$ it will always remain so, for, by eqn. (4), the vector of acceleration is perpendicular to the vector $\mathbf{H}$ as well as to the vector $\mathbf{v}$. Hence, in the special case of a constant and homogeneous transverse magnetic field, sin $(\mathbf{v}, \mathbf{H})$ is always equal to unity ; and it follows from eqns. (7), (8) and (9) that in this special case

$$(10) \qquad\qquad r = \text{const.}$$

But the constancy of the radius of curvature indicates that the body must describe a *circular path in a plane at right angles to the magnetic lines of force* ; and from eqns. (8) and (9) we find the value of the radius of the circle to be

$$(11) \qquad\qquad r = \frac{vc}{\gamma H}.$$

§ 72. Electromagnetic Mass.

The idea of the production of a *magnetic field* by every electrically charged body in motion leads to a very remarkable conclusion, to which attention was first called by *Joseph John Thomson* in 1881, and which has given an entirely new significance to one of the most important fundamental conceptions of physics—the conception of *mass*.

From Biot and Savart's law we can at once calculate the

energy of the magnetic field produced by a moving electric charge. To do this we first substitute the product of current density and elementary volume for the product $I \, ds$ in § 61 eqn. (3). Then, on applying § 70 eqn. (1) and integrating over the volume of the charge, we find for the magnetic strength of field

$$(1) \qquad H = \frac{Qv}{cr^2} \sin \theta \,,$$

where θ denotes the angle included between the direction of motion and the radius-vector $\mathbf{r}$ drawn to any given field-point. Now the magnetic energy of the elementary volume $(d\tau)$ enclosing the field-point is, by § 55 eqn. (8),

$$(2) \qquad dW = \frac{H^2}{8\pi} \, d\tau \,.$$

Hence we have for the total energy of the produced magnetic field

$$(3) \qquad W = \frac{Q^2 v^2}{8\pi c^2} \int \frac{\sin^2 \theta \, d\tau}{r^4} \,.$$

The integral in eqn. (3), however, will always be completely independent of the charge and the velocity, and thus we see that the *energy of the magnetic field* is *directly proportional to the square of the velocity of the moving charge.* This energy will be zero when the charge is at rest, but it will differ from zero when the charge is in motion. *In the process of passing from a state of rest to one of motion energy must be expended*, the amount being proportional to the square of the velocity produced.

Now an expenditure of energy is necessary in order to set an *inertial mass* in motion, the amount being given by half the product of the mass and the square of the velocity imparted. Therefore, as J. J. Thomson pointed out, we must ascribe to every electric charge a so-called *electromagnetic mass* which, in accordance with eqn. (3), is determined by the expression

$$(4) \qquad m_e = \frac{Q^2}{4\pi c^2} \int \frac{\sin^2 \theta \, d\tau}{r^4} \,,$$

the integral being taken over the whole of the magnetic field produced by the charge.

We will now calculate the electromagnetic mass in the particular case of a *charged spherical surface*, within which we assume no magnetic field to exist ; the integration is then to be carried out throughout space, with the exception of

the interior of the sphere. We may for convenience suppose the plane of Fig. 53 to be bisected by the straight line which passes through the centre of the sphere, and has the direction of the velocity. We may then imagine the upper half of the plane to rotate once round this line as axis. Now, if we draw a rectangular surface-element with sides dr and $r\, d\theta$, it will describe a circle whose circumference is $2\pi r$ $\sin \theta$. Hence the volume of the body resulting from the rotation of the surface-element about the axis will be

$$(5) \qquad d\tau = 2\pi r^2 \sin \theta\, dr\, d\theta .$$

After inserting this value in the integral in eqn. (4), we have to integrate over all the values of r from a to ∞ (where a is
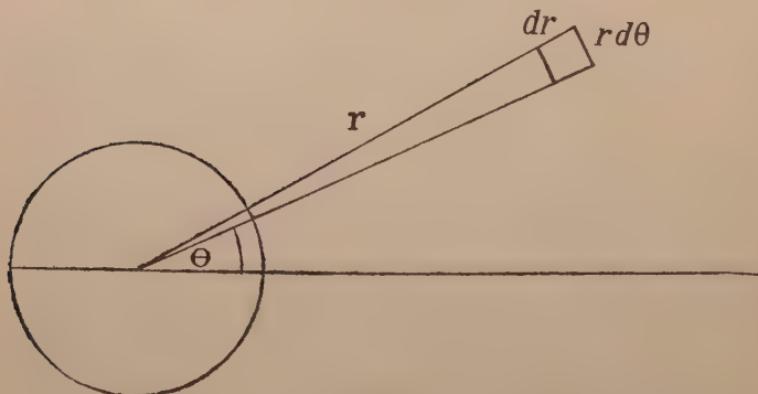
Fig. 53.

the radius of the sphere), and over all values of θ from 0 to π (only values between 0 and π are possible, because the *whole* of space results from one complete rotation of the *upper* half of the plane of the figure). We thus obtain for a charged spherical surface

$$(6) \qquad m_e = \frac{Q^2}{2c^2} \int_a^\infty \int_0^\pi \frac{\sin^3 \theta}{r^2}\, dr\, d\theta .$$

But

$$(7) \qquad \left\{ \begin{aligned} \int_0^\pi \sin^3 \theta\, d\theta &= -\int_0^\pi (1 - \cos^2 \theta)\, d(\cos \theta) \\ &= \left| \cos \theta \right|_\pi^0 - \left| \frac{\cos^3 \theta}{3} \right|_\pi^0 = \frac{4}{3}, \end{aligned} \right.$$

and

$$(8) \qquad \int_a^\infty \frac{dr}{r^2} = \frac{1}{a}.$$

Hence eqn. (6) becomes

(9) $$m_e = \frac{2}{3}\frac{Q^2}{c^2 a}.$$

Thus the electromagnetic mass of a spherical charge proves to be *proportional directly to the square of the charge* and *inversely to the radius of the sphere*.[1]

§ 73.   The Dielectric.

In 1838 *Faraday* made a discovery which has proved to be of extraordinary importance in the further development of theoretical physics, *viz.*, that the *mutual action between two electrically charged bodies depends upon the nature of the intervening medium*.[1]   He found that the *capacity of a spherical condenser* changes when the space between the two

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[1] If we consider two equal spherical charges of the same radius moving with the same velocity in the same direction, and if, by reason of their motion, they create magnetic fields whose strengths at any given field-point are $\mathbf{H_1}$ and $\mathbf{H_2}$, then the square of the *resultant* strength of field at this point is given by

(10) $$H^2 = H_1{}^2 + H_2{}^2 + 2H_1 H_2 \cos(\mathbf{H_1}, \mathbf{H_2}).$$

The resultant electromagnetic mass will therefore not be equal simply to the sum of the two individual electromagnetic masses, but there will be an additional term in the form of a *mutual electromagnetic mass*.   *Silberstein* [*Phys. Zeits.* **12** (1911), pp. 87–91] has calculated that the mutual electromagnetic mass of two identical and completely separate spherical charges of radius a, whose centres are at a distance d apart, is given by

(11) $$m_{12} = m \frac{a}{2d}\left(5 - \frac{2a^2}{d^2}\right),$$

where m is the individual electromagnetic mass.   If the distance between the centres is equal to ten times the radius, the mutual mass is approximately equal to $m/4$; if d is equal to $100a$, the mutual mass is still equal to about one-fortieth of the individual mass.   When the two charges touch one another, so that d is equal to $2a$, the mutual electromagnetic mass is equal to $9/8$ times the individual mass, and the total electromagnetic mass is thus equal to $3\frac{1}{8}m$.   If d vanishes [eqn. (11) is then no longer applicable], *i.e.*, if the centres of the two charges coincide, the total electromagnetic mass becomes equal to four times the individual mass.   In that case the combination of the two charges really represents one single charge of double magnitude, but with the same radius, so that the total mass will be four times as great as that of either charge, as follows from eqn. (9).   If we consider a charge uniformly distributed *throughout the volume*, instead of over the surface of a sphere, the numerical factor $2/3$ in eqn. (9) will be replaced by the factor $4/5$.

[1] This experiment of Faraday's owes its great historical significance to the fact that to it was due the displacement of the earlier, so-called *theory of action at a distance* of electrical phenomena, by a *theory of action in the surrounding medium (proximate action)*.   From the point of view of the former theory, the electrostatic *field* produced by a charge was a *purely geometrical construction* without physical meaning, at least so long as no second charge was present. The newer theory, however, always ascribes a *real physical significance to the electric field*.   Every charge acts, in the first place, upon its immediate surroundings ; and it is only *through the medium* of these that it acts upon other charges at a distance.   The hypothesis of displacement currents necessitates an extension of this conception, as it were, to space itself.

concentric spheres is filled with a medium other than air. For example, the capacity becomes twice as large when the intervening layer consists of paraffin, three times as large when it consists of shellac, six times as large for glass, and some eighty times as large for water. The number giving the ratio in which the capacity is increased is known as the *dielectric constant* relative to air of the medium in question.

More exact experiments have shown that the capacity of a spherical condenser decreases when the air between the two shells is removed. If C_0 be the limiting value approached by the capacity as the pressure is diminished, and C^* the capacity when the intermediate space is filled with air at normal pressure, then the ratio C^*/C_0 is called the dielectric constant of air relative to a vacuum, or simply the dielectric constant of air. We can thus obtain the dielectric constant of any given medium relative to a vacuum by multiplying the constant relative to air by the quotient C^*/C_0. As a matter of fact, the dielectric constant of air scarcely differs from unity, for its value is 1·00059.

In the dependency of electrical actions upon the intervening medium, thus experimentally demonstrated, we must at the present time perceive a convincing proof of the accuracy of the modern conception, which regards all matter as composed of electrically charged particles. Moreover, it is only from this point of view that this dependency can be *explained* in a theoretically satisfactory manner. A simpler way, however, suffices for the *phenomenological description* of this dependency ; it does not demand a more detailed examination of modern conceptions, and had actually been utilized by *Maxwell* long before these conceptions were established.

Maxwell's fundamental assumption was to the effect that, in an intervening medium (generally termed the *dielectric*) of dielectric constant ϵ, the *electric field-strength* is *reduced* in the ratio $1 : \epsilon$ relatively to the value which it would have at the same point *in vacuo*, under otherwise identical conditions. Denoting the latter value by $\mathbf{E}_0$, we therefore have

$$(1) \qquad \mathbf{E} = \frac{1}{\epsilon} \mathbf{E}_0.$$

Correspondingly, the magnitude of the Coulomb force between two charges e and e' at a distance r apart is given by

$$(2) \qquad K = \frac{ee'}{\epsilon r^2}.$$

Provided that the dielectric be *homogeneous*, *i.e.*, that in it ϵ has everywhere the same value, the *potential* naturally obeys the relation

$$(3) \qquad \Psi = \frac{1}{\epsilon} \Psi_0,$$

where Ψ_0 denotes the potential *in vacuo* under otherwise identical circumstances.[2] It follows from eqn. (3) that, with an equal charge, the capacity of a condenser (the quotient of charge and potential) will actually appear to be increased in the ratio $\epsilon : 1$, as already discovered by Faraday.

Now we know from § 50 eqn. (14) that in free space every charge, multiplied by 4π, can be equated to the flux of electric force through any arbitrary surface enclosing the charge. In the case of a homogeneous dielectric we therefore have

$$(4) \qquad 4\pi e = \epsilon \int E_n \, df.$$

It follows from § 66 eqns. (1—4) that the *density of the displacement current* in a homogeneous dielectric is

$$(5) \qquad \mathbf{g} = \frac{\epsilon}{4\pi} \frac{\partial \mathbf{E}}{\partial t}.$$

Let us next consider the *interface between two dielectrics*, first at a point in the electric field where the *lines of force* lie at *right angles to the interface*. It is perfectly clear that these lines of force will not change their direction in passing from the one medium to the other, for, in the case of normal incidence, there is no reason why they should deviate to the right-hand rather than to the left-hand side. In accordance with eqn. (1) we thus have on the one side of the interface

$$(6) \qquad E = \frac{E_0}{\epsilon},$$

while on the other side, where we suppose the dielectric constant to be ϵ', we have

$$(7) \qquad E' = \frac{E_0}{\epsilon'} = E \frac{\epsilon}{\epsilon'}.$$

Since E' thus differs from E when the values of the two

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[2] By eqn. (1),

$$\mathbf{E} = -\frac{1}{\epsilon} \operatorname{grad} \Psi_0;$$

and hence, when ϵ is constant, $\mathbf{E}$ may be equated to the negative gradient of (Ψ_0/ϵ).

dielectric constants are different, the interface behaves as if it possessed a *surface charge* [according to § 52 eqn. (1)].

Further, in the case of any *arbitrary angle of incidence of the electric lines of force*, the *normal component of the strength of field* will undergo the *sudden change* described by eqn. (7). On the other hand, the *tangential component* will be *continuous in crossing the interface*, just as for any charged surface—as follows from § 52 eqn. (4). Hence, distinguishing the normal and the tangential components by the indices n and t, we have the quite general relations

$$(8) \qquad E_n' = E_n \frac{\epsilon}{\epsilon'}$$

and [3]

$$(9) \qquad \mathbf{E}_t' = \mathbf{E}_t .$$

Let φ and φ' be the angles included between the directions of the force in the two media and the *normal* to the interface ; then

$$(10) \qquad E_t = E_n \tan \varphi , \qquad E_t' = E_n' \tan \varphi' .$$

From eqn. (9) we hence obtain the relation

$$(11) \qquad \tan \varphi : \tan \varphi' = E_n' : E_n ,$$

or, by eqn. (8),

$$(12) \qquad \tan \varphi : \tan \varphi' = \epsilon : \epsilon' .$$

This ratio is an expression of the *law* according to which the *direction of electric force is refracted* at the interface between two media.

§ 74. Magnetic Permeability.

Just as in the case of the fundamental law of electrostatics, the *law of the magnetic attraction between two closed currents* [1] requires but slight *modification*, if an approximately correct allowance is to be made for the *influence of the intervening medium* on electromagnetic phenomena. We have to introduce, besides the dielectric constant, a *second* constant characteristic of the medium, and known as the *magnetic permeability*, generally denoted by μ.

Thus the modified law of the magnetic attraction between

[3] It must be remembered that eqn. (9) expresses the identity, not only of the magnitudes of the tangential components, but also of their directions. In the case of the normal components this is self-evident, for there is obviously only one direction normal to the interface. The tangential components, however, only define a plane within which their directions must lie.

[1] *Vide* § 60.

closed currents might be expressed in the following form :—
In any given medium, two closed currents exert upon each
other a mechanical force which is just as large as the elec-
trostatic force between two shells bounded by the currents
and situated in a medium whose dielectric constant is μ,
provided that the density of moment of either shell be made
equal to the current, in electromagnetic measure, multiplied
by μ.

When the intervening medium is taken into account, it
therefore follows that, on the one hand, the *density of
moment*

$$(1) \qquad\qquad \chi = \mu \frac{I}{c} \; ;$$

and, on the other hand, the *magnetic strength of field*

$$(2) \qquad\qquad \mathbf{H} = \frac{1}{\mu} \mathbf{H}_0 \, ,$$

where $\mathbf{H}_0$ is the value of the field *in vacuo*, under otherwise
identical circumstances. Hence we obtain for the magnitude
of the force of attraction or repulsion between two quantities
of magnetism m and m', placed at a distance r apart,

$$(3) \qquad\qquad K = \frac{mm'}{\mu r^2} .$$

Denoting the curve potential of the closed current by $\mathbf{P}$,
we see that both it and its rotation, as purely geometrical
quantities, are obviously entirely independent of the mag-
netic permeability. By eqn. (2) [as well as by § 54 eqn. (11)],
however, the magnetic strength of field produced by a closed
current at any arbitrary field-point is given by

$$(4) \qquad\qquad \mathbf{H} = \frac{1}{\mu} \chi \operatorname{rot}_a \mathbf{P} .$$

On substituting for χ its value by eqn. (1), we see that the
magnetic field-strength produced by a current at any given
field-point is *independent of the magnetic permeability of the
surrounding medium.* Hence *Biot and Savart's law*, and thus
also the *first principal equation of the electromagnetic field*
[§ 61 eqn. (8)], retain the *same form* for any arbitrary medium.
In accordance with § 67 eqn. (2) and § 73 eqn. (5) Maxwell's
fourth equation therefore becomes

$$(5) \qquad\qquad \operatorname{rot} \mathbf{H} = \frac{4\pi}{c} \lambda \mathbf{E} + \frac{\epsilon}{c} \frac{\partial \mathbf{E}}{\partial t} ,$$

where λ again denotes the specific conductivity ; if no conduction current be present, the first term on the right-hand side naturally vanishes.

It follows from eqn. (3) that the mechanical potential energy of two currents relative to each other must be proportional to the expression $\chi\chi'/\mu$, and hence, by eqn. (1), to μ. According to § 63 the same relation must hold good for the induced electromotive force, which will be increased in the ratio $\mu : 1$ in a medium whose magnetic permeability is μ, as compared with its value *in vacuo* under otherwise identical circumstances. Consequently, when the surrounding medium is taken into account, the *second principal equation of the electromagnetic field* (*i.e.*, Maxwell's third equation) assumes the form

$$(6) \qquad \text{rot } \mathbf{E} = -\frac{\mu}{c}\frac{\partial \mathbf{H}}{\partial t}.$$

Whilst the dielectric constant of *every* substance is greater than unity, this is not the case with the magnetic permeability, as experience shows. It may be greater or smaller than unity ; in the first case the medium is said to be *paramagnetic*, in the second *diamagnetic*.[2]

§ 75. The Propagation of Electromagnetic Waves in the Dielectric.

As the considerations of the two previous sections have shown, the influence of the dielectric may be more or less correctly described by introducing the dielectric constant and the magnetic permeability into the various equations. In doing this, however, we assume that no possible, invisible electric charges are present in the intervening medium. If we consider a medium which, in this sense, is *free from charges*, and which does not conduct electricity—*i.e.*, an *insulator*—we see that on the one hand the equations of § 69 (1) hold good, whilst on the other hand Maxwell's third and fourth equations become, in accordance with § 74 eqns. (6) and (5),

$$(1) \qquad \text{rot } \mathbf{E} = -\frac{\mu}{c}\frac{\partial \mathbf{H}}{\partial t}, \qquad \text{rot } \mathbf{H} = \frac{\epsilon}{c}\frac{\partial \mathbf{E}}{\partial t}.$$

[2] The product of the magnetic strength of field and the magnetic permeability is often known as the *magnetic induction*, while the product of the electric strength of field and the dielectric constant (always divided by 4π) is called the *dielectric displacement*.

Hence [*cf.* § 69 eqns. (2) and (3)],

$$(2) \quad \begin{cases} \text{rot rot } \mathbf{E} = - \nabla^2\mathbf{E} = -\dfrac{\mu}{c} \cdot \dfrac{\partial}{\partial t} (\text{rot } \mathbf{H}), \\[2mm] \text{rot rot } \mathbf{H} = - \nabla^2\mathbf{H} = \dfrac{\epsilon}{c} \dfrac{\partial}{\partial t} (\text{rot } \mathbf{E}), \end{cases}$$

whence we obtain

$$(3) \qquad \nabla^2\mathbf{E} = \frac{\epsilon\mu}{c^2} \frac{\partial^2\mathbf{E}}{\partial t^2}, \qquad \nabla^2\mathbf{H} = \frac{\epsilon\mu}{c^2} \frac{\partial^2\mathbf{H}}{\partial t^2}.$$

Thus *electromagnetic waves* are propagated *through the dielectric* with a *velocity*

$$(4) \qquad v = \frac{c}{\sqrt{\epsilon\mu}}.$$

The above value of v takes the place of c in § 69 eqn. (4). Moreover, § 69 eqns. (5) become, when the influence of the dielectric is taken into account,

$$(5) \quad \frac{\partial H_x}{\partial t} = 0, \qquad \frac{\mu}{c}\frac{\partial H_y}{\partial t} = \frac{\partial E_z}{\partial x}, \qquad \frac{\mu}{c}\frac{\partial H_z}{\partial t} = -\frac{\partial E_y}{\partial x}.$$

On substituting for E_y and E_z their values as given by § 69 eqns. (4), replacing therein c by v in accordance with what we have just said, we find

$$(6) \quad \begin{cases} \dfrac{\mu}{c} \dfrac{\partial H_y}{\partial t} = Ba \cos [a\, (x - vt)\,], \\[2mm] \dfrac{\mu}{c} \dfrac{\partial H_z}{\partial t} = - Aa \cos [a\, (x - vt) + \delta]\,; \end{cases}$$

where the constants A, B, a, and δ have the same meaning as in § 69.

By integrating these equations, the functions independent of the time being omitted, we obtain the corresponding relations to § 69 eqns. (7) :

$$(7) \qquad \frac{\mu}{c} H_y = -\frac{1}{v} E_z, \qquad \frac{\mu}{c} H_z = \frac{1}{v} E_y,$$

or, by eqn. (4),

$$(8) \qquad H_y = - \sqrt{\frac{\epsilon}{\mu}} E_z, \qquad H_z = \sqrt{\frac{\epsilon}{\mu}} E_y.$$

Since we have chosen the direction of propagation of the waves as x-axis (just as in § 69), E_x naturally vanishes, as does H_x [in accordance with the first of eqns. (5)]. Thus, if

f be the unit-vector lying in the direction of propagation of the waves,[1] we may also write eqn. (8) in the vectorial form

(9)
$$\mathbf{H} = \sqrt{\frac{\epsilon}{\mu}}\,[\mathbf{f}\mathbf{E}].$$

[1] The components of the unit-vector **f** are given by the relations
$$f_x = 1, \quad f_y = {}_z = 0.$$

THEORY OF LIGHT

§ 76. The Electromagnetic Nature of Light.

As was shown in one of the previous sections, the fundamental assumptions of *Maxwell's theory* lead to the *theoretical possibility of electromagnetic waves*, and we have found as characteristic properties of the latter their *transverse nature* and the value of their *velocity of propagation*. It follows from our theoretical deductions that in *free space* this *velocity* must be *equal to the electromagnetic constant*, whereas in a *dielectric* a simple relation exists between the velocity and the constants characteristic of the dielectric.

Now *light* is found to exhibit, besides the general properties of all undulatory processes, these special properties of electromagnetic waves. Optical phenomena are thus obviously based on electromagnetic undulatory processes ; and the most convincing proof of the accuracy of this conception (which was established by *Maxwell* in 1873) is afforded by the *experimentally determined complete agreement between the electromagnetic constant and the velocity of light*.

The value of the *velocity of light* was first determined by *Roemer* in 1676, and that with very fair accuracy. Roemer discovered that the *occultations of Jupiter's satellites* are apparently retarded or advanced according as the distance of Jupiter from the earth is greater or smaller. Even at that time he recognized as the cause of this phenomenon the fact that, the further away Jupiter is, the longer the light from its satellites requires to reach the earth. From the known astronomical distances and from the observed apparent retardation of the occultations, Roemer was able to compute the velocity of light remarkably accurately.[1] Half a century later, in 1727, the astronomer *Bradley* arrived at the same value of the velocity of light by quite another method, based

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[1] Roemer observed that the difference of 40 million miles in the distance between Jupiter and the earth at opposition from that at conjunction is traversed by light in 1,000 seconds. This result is accurate to about 1 per cent.

on the *aberration of light* (a phenomenon that will be further discussed in a later chapter), a consequence of which is an annual, apparent, periodic motion of the fixed stars. Bradley recognized from this phenomenon, in complete agreement with Roemer, that the velocity of light must be ten thousand times the velocity of the earth in its orbit, which had been calculated by astronomers to be 30 km. per second.

In 1849, *Fizeau* succeeded in accurately determining the velocity of light from observations *on the earth*. In his experiments, a pencil of light from a luminous source was reflected by a mirror in such a way that, both on its outward and on its return journey, it had to pass through the openings between the teeth of a *rapidly rotating toothed wheel*. If the speed of revolution be continuously increased, an observer will see the pencil disappear at a certain moment, *viz.*, when the wheel revolves through an amount exactly equal to half the distance between two adjacent teeth, during the time that the ray twice covers the distance between wheel and mirror. Fizeau's measurements also led to the same value of the velocity of light as the astronomical observations of Roemer and Bradley ; and the value derived by these three fundamentally different methods agreed exactly with that obtained by *Weber* and *Kohlrausch* in 1856 from their purely electrical measurements of the *electromagnetic constant* (*cf.* § 61).[2]

The perfecting of Fizeau's method rendered possible a determination of the velocity of light with an accuracy which only left room for a possible error of perhaps one ten-thousandth part of the measured value. At the present time, the exact value of the velocity of light is accepted as being

(1) $$c = 2 \cdot 9990 \times 10^{10} \text{ cm. sec.}^{-1},$$

for which, with a very great approximation that is sufficient for almost all purposes, we may substitute the value $3 \cdot 00 \times 10^{10}$ cm./sec. In recent years the electromagnetic constant has also been determined with an accuracy of about one in a thousand by purely electrical methods, the mean value amounting to $3 \cdot 00 \times 10^{10}$ cm./sec. Hence there is actually a complete agreement between the results of the optical and the electrical measurements.

Now that the electromagnetic character of light has been proved, theoretical physics is faced with the problem of

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[2] Weber and Kohlrausch found a value of 311,000 km./sec. for the electromagnetic constant.

deriving from the properties of electromagnetic waves all the phenomena of *experimental optics.* Even before the recognition of the electromagnetic nature of light, however, theoretical optics had attained a high degree of perfection ; for many optical phenomena find an explanation in the *general undulatory character of light,* without our necessarily having recourse to the special knowledge that, in light, we are dealing with electromagnetic waves.

In the 17th century it first became clear to some physicists that light must be based on an undulatory process. Such an assumption was, it is true, in no way demanded by the fundamental phenomena upon which, to a certain extent even in ancient times, *geometrical optics* had been built up. These fundamental phenomena, indeed, such as the *rectilinear propagation of light,* the *equality of the angles of incidence and reflection* (known to the ancients), and the law of *refraction* discovered by *Snellius* in 1620, could be far more easily explained by the assumption that light was a substance consisting of small inertial particles, than from the point of view of a wave theory.

During the second half of the 17th century, however, physicists became acquainted with a series of new optical phenomena which gradually revealed to them the *undulatory nature of light.* These phenomena included the *diffraction of light,* discovered by *Grimaldi* ; *double refraction,* discovered by *Bartholinus* ; the observation of the *colours of thin films,* first noted by *Hooke* ; and the so-called *Newton's rings.* The obvious consequence of such a series of newly discovered fundamental phenomena (to which must be added that of *dispersion,* also discovered by *Newton*) was to lead physicists to form definite conceptions of the nature of light. Thus, almost simultaneously (about 1670), the two leading physicists, *Newton* and *Huygens,* came to the important conclusion that *light* must be a *periodic process both as regards time and space.*

Huygens gave a sharper definition to the periodicity of light than Newton, by referring light to *processes of mechanical vibrations.* Inasmuch, however, as light is transmitted from the sun to the earth through a space which is obviously free from ponderable matter, for it would also be difficult otherwise to explain the immense value of the velocity of light, the latter could not be regarded as a wave motion of any ordinary substance. Huygens had thus no alternative but to make a hypothetical " *æther* " the carrier of the waves of light, such as had been previously conceived by other thinkers

for other reasons.[3] As in solid or liquid bodies, so also in the æther, according to Huygens' idea, it should be possible to have *elastic waves* which we perceive as light.

Science owes a great and lasting debt to Newton and Huygens, for they were the first to recognize the space-time periodicity and thus (in the wider sense of the word) the wave nature of light. The fact that optical waves were regarded in particular as mechanical ones is due, historically, to the tendency in physics at that time of *reducing everything to mechanics*, but this was in no way necessary. Newton's and Huygens' recognition of the space-time periodicity of light would have completely sufficed for the great advances which were actually made in theoretical optics up to the time of Maxwell. This would have been none the less true even if the question as to the true nature of these doubly periodic processes had been left unsolved, and a premature decision had not been made in favour of the special assumption of the mechanico-elastic nature of optical processes.[4]

With the help of the wave theory of light Huygens had already been able to give a satisfactory explanation of the

[3] The æther played a great part even in the natural philosophy of Descartes.
[4] During the 18th century and the first two decades of the 19th century the *wave* and the *emission theory of light* vied with each other for precedence. The former regarded light as being mechanical waves, whilst the latter regarded it as matter in the form of very small particles. *Newton* certainly laid the foundation for the emission theory, although he intentionally avoided taking a stand for any definite hypothesis. But we should be quite unjustified were we to disparage Newton's services to theoretical optics on that account. Newton's important discovery of the periodicity of light is independent of all special conceptions ; indeed, from the modern point of view, we must actually give preference to a theory which referred to periodicity only in a general sense, and not specially to mechanical waves.—In point of fact, strong support was afforded to the emission theory by the rectilinear propagation of rays of light, for this followed at once from the inertia of the light particles. The emission theory explained reflection and refraction by the assumption of forces of repulsion and attraction exerted by ponderable matter upon the light corpuscles. It referred the different colours to corpuscles of different sizes, the red ones being the largest, and the violet ones the smallest. Moreover, as the forces exerted by matter upon the corpuscles must obviously depend upon the size of the latter, this assumption gave a ready explanation for the dispersion of light. Very complicated assumptions were necessary to explain Newton's rings and the colours of thin films in terms of the emission theory, and even more complicated were those required to interpret the phenomenon of double refraction. Yet an explanation was always possible on the ground of Newton's fundamental assumption, that certain properties are periodically repeated at equal intervals along a ray of light ; and, in fact, Newton had already recognized that these intervals must be shortest for violet light, and longest for red light.— It is true that the wave theory also had its weak sides. To it the rectilinear propagation of light long remained a complete mystery. It completely failed in the case of the dispersion of light ; indeed, an explanation for that phenomenon was not forthcoming until Maxwell's theory had been supplemented by the electron hypothesis.

complicated phenomena of double refraction ; and *Young,* when he rescued the wave theory from almost complete oblivion early in the 19th century, at the same time supplementing it by the principle of interference, was also able to explain Newton's rings, and he actually made the first determinations of the wave-lengths of light by their aid. In 1816 *Fresnel* succeeded also in solving the apparent contradiction between the wave hypothesis and the linear propagation of light, and thereby found an explanation for the phenomena of diffraction. In the same year Fresnel finally disproved the corpuscular theory of light by means of his famous mirror-experiment on interference (which we shall discuss more fully later on) ; this clearly showed that the addition of light to light can produce darkness.

Between 1808 and 1820 a new group of optical phenomena were discovered, which are collectively known as the *phenomena of polarization.* These discoveries made it especially clear to physicists that the principle of the space-time periodicity of light was capable of giving an extremely simple explanation even for quite complicated optical phenomena. The phenomena of polarization found a very simple explanation on the optical wave hypothesis, but not until this had received an important and fundamental extension in the particular assumption of the *transverse nature of light waves.*[5] As we shall show in greater detail later, this conception is due to *Young,* and it forms the basis of *Fresnel's theory of light,* which is founded on the concept of transverse elastic waves in the æther.

Successful though this theory was in the explanation of the complicated phenomena of polarization, yet the actual transverse nature of the vibrations of light presented to it an insoluble riddle. For, in accordance with the theory of elasticity (as we saw in § 49), *only longitudinal waves* could occur in the æther, were it a *fluid.* Transverse waves, in addition to longitudinal ones, are only possible if we regard the æther as *solid* ; whilst, if we desire entirely to exclude longitudinal waves, it follows from § 49 that we have no option but to consider the æther as being also *incompressible.* It is difficult, however, to make this idea consistent with the necessary assumption that the æther offers no resistance to the motion of the heavenly bodies.

All these difficulties vanished with the foundation of the *electromagnetic* theory of light, in which the transverse nature

[5] Huygens had supposed light waves to be longitudinal, like waves of sound.

of light waves actually follows with mathematical certainty from the fundamental equations of the theory. A further great advantage of the electromagnetic theory of light lay in the fact that the phenomena of the reflection and the refraction of light could be explained by Maxwell's theory, as we shall later demonstrate in greater detail, without drawing upon any further assumptions. Contrasted with this, these phenomena had been explicable on Fresnel's elasticity theory of light only with the help of several highly artificial hypotheses, expressly devised for the purpose.

As we have already shown, the agreement between the velocity of light and the electromagnetic constant gave the strongest support in favour of Maxwell's theory. It also follows from the electromagnetic theory of light, however, that a definite relation exists between the dielectric constant of a medium and a constant which characterizes the optical behaviour of the medium (*viz.*, the refractive index) : we shall examine this relation more in detail later on. This relation derived from the theory has also been confirmed by accurate measurements.

The most convincing proof of the correctness of the electromagnetic theory of light, however, resulted from the famous experiments carried out by *Hertz* in 1888. These experiments were based on the *discharge oscillations of condensers*, which we have discussed in a previous section (§ 65). About 1860, by means of a very rapidly rotating mirror, *Feddersen* had drawn out the photographic image of the spark discharge of a Leyden jar, and had thus been able to determine the number of oscillations in a second as ranging from about 10,000 to 1,000,000. If we assume, in accordance with Maxwell's theory, that these oscillations lead to the formation of electromagnetic waves which are propagated with the velocity of light, we find that the wave-lengths must be of the order of magnitude of kilometres ; and, of course, we cannot make experimental observations with waves of such length.

On the other hand, however, the formula giving the period of condenser oscillations [§ 65 eqn. (36)] shows us how the period, and with it the *wave-length*, can be *reduced*. To do this, we have to diminish the capacity and the inductance of the circuit as far as is practicable. Hertz accomplished this by means of different experimental artifices, which here we cannot do more than mention. (For example, he replaced the Leyden jar by two brass plates connected by a thick copper

wire, a spark-gap between brass balls being inserted in the middle of the wire.) In this way Hertz was able to reduce the wave-lengths of the electric discharge waves to a few metres, and later even to a few decimetres ; with waves of this length he was able to experiment conveniently. He thus succeeded in proving that these *waves*, which were *produced by purely electrical means*, possessed the *same properties as waves of light* ; like light waves, they could be reflected, refracted, diffracted, polarized, and made to interfere ; moreover, they were propagated with the velocity of light.[6]

As was first shown by a method to be described in § 79, the wave-lengths of the light *perceptible to the human eye* lie between about 4×10^{-5} cm. (corresponding to violet light) and about 8×10^{-5} cm. (corresponding to red light). That is to say, expressing wave-lengths in the usual units, they lie between about 4,000 and 8,000 *Ångström units* (Å.U.), for

(1) $1 \text{ Å.U.} = 10^{-8}$ cm.

The values of the wave-lengths of the other colours in the rainbow lie between these two limiting values in the same order as in the rainbow itself. To these wave-lengths correspond *frequencies of between* 400 *and* 800×10^{12} *a second*, the value, of course, being smallest in the case of red, and largest in the case of violet light. Since the frequency is approximately twice as large for violet as for red light, it is commonly said—to borrow an expression from acoustics— that *visible light extends over an octave*.

The *infra-red rays*, which were first discovered from their thermal effects by *Herschel* in 1800, comprise, at the present stage of research, a zone of rather more than *eight octaves*. The longest wave-length yet demonstrated by an optical method is one of 0·343 mm., whilst, in the production of very short electric discharge waves, a wave-length of only 0·22 mm. has already been attained.[7]

[6] Hertz calculated the velocity of propagation of the discharge waves from the distances between the nodes of stationary waves ; he found a value of 280,000 km./sec.

[7] Up to the end of the nineteenth century, investigations of the infra-red region extended to wave-lengths of not more than 10,000 Å.U. In 1898 *Rubens* succeeded in isolating infra-red rays of larger wave-length by means of the so-called *residual ray* method. A large number of substances possess the property of so-called *selective* reflection and absorption, in that they reflect and absorb rays of certain wave-length to an exceptional extent. Hence, if light be allowed to travel several times to and fro between mirrors made of such a substance, it is possible to isolate fairly monochromatic rays of the wave-length concerned, in the form of so-called residual rays. Using sylvine, Rubens obtained in this way rays of 0·061 mm. wave-length. In 1910 he detected rays of still larger wave-

The *ultra-violet rays*, discovered in 1801 by *Ritter* through their chemical effects, extend, in the present state of experimental physics, over a range of some *six octaves* down to a wave-length of about 80 Å.U.[8] A gap of some two octaves, within which only indirect measurements are at the moment possible, still separates the region of ultra-violet rays from that of the extremely short electromagnetic waves known as *Röntgen rays*. Direct measurements on these rays (to be described at length in a subsequent chapter) cover the zone between 20 and 0·05 Å.U., or more than eight octaves.

§ 77. Natural and Polarized Light.

Since (by § 39) all *vector waves* are founded on *elliptical vector vibrations* which, in special instances, may be *linear* or *circular*, we may expect to find that the properties of a ray of light will differ in different directions perpendicular to its direction of propagation.[1] Now *lateral differences* of this kind were first actually observed by *Huygens* (about 1680) in *rays of light* which were *doubly refracted* by a crystal of calcite ; and they were observed in 1808 by *Malus* in rays of *reflected*, and shortly afterwards in rays of *refracted* light.[2]

length by another method. This method was based on the fact that the refractive index of quartz has entirely different values, according as the wave-lengths concerned are larger or smaller than 0·08 mm., the latter value corresponding to the selective reflection of quartz. An arrangement of quartz lenses thus presents the possibility of the isolation of rays of extremely large wave-length, and a suitable source for the emission of such rays was found by Rubens and his collaborators in the mercury lamp. From its radiation they were able to separate two groups, one of 0·218 mm. and the other of 0·343 mm. wave-length. —The smallest wave-length hitherto detected in electric discharge waves is one of 0·22 mm., obtained in 1923 by *Nichols* and *Tear :* up till then the shortest known wave had been one about 3 *mm.* in length attained by *Lampa* in 1897.

[8] Investigations of ultra-violet rays with the help of photographic plates sufficed only to reach a wave-length of some 2,000 Å.U. A further advance was apparently hindered by the very strong absorption of rays of still smaller wave-length by air, and also by the binding medium on the photographic plate. These difficulties were first overcome by *Schumann* who, in 1892, reached 1,270 Å.U. Research on the more extreme regions in the ultra-violet have been carried out principally by Lyman.

[1] In the case of circular vibrations, we may at least distinguish a certain sense of rotation in the ray, corresponding to the contrast between opposite circular oscillations.

[2] Huygens showed that, under certain circumstances, the ordinary and extraordinary rays arising from double refraction are, in their turn, refracted, not doubly but only singly—in contrast to rays of natural light—on passing through a (second) crystal of calcite (even when their direction is not that of the principal optical axis). Malus proved that, with a particular experimental arrangement, a ray of light may be extinguished by reflection at two mutually perpendicular mirrors. In this connection *cf.* § 82.

Malus also recognized that such lateral differences are only to be observed in rays of light which have been subjected to one or another *optical process*, whilst they are *absent* in rays coming *directly from a luminous source*.

These important experimental facts led Malus to distinguish between *natural* and *polarized* light. We define natural light as that which comes directly from a luminous source, and polarized light as that which, in consequence of an optical process to which it has been subjected on its way from the source, exhibits lateral differences. The expression " polarized " finds a historical explanation in the fact that, in a ray of light of this kind, certain directions normal to the direction of propagation are emphasized, in much the same way as in a magnetized body the axis joining the poles takes up a unique position as compared with all other directions.

Thus, although natural light does not exhibit any lateral differences, we can produce from it, by means of double refraction or reflection or even simple refraction, light which does exhibit these differences. As it is generally expressed, we are able to *polarize* natural light. This fact finds its simplest explanation in a hypothesis of *Fresnel's*, according to which natural light is also polarized at any instant, but the *state of vibration changes so frequently* during the small interval of time required for observation that, as a whole, no one direction is in any way given preference to any other. As we shall show later, visible light executes hundreds of billions of vibrations in a second ; hence even millions of vibrations could occur in succession always with the same state of vibration, and yet the state of vibration could still change millions of times a second. Thus *no definite state of polarization* could, in fact, be *detected in natural light*, because the constancy of such a state would only hold over *intervals of time* which were *too small for observation*.

When the light vibrations are linear, the light is said to be *plane polarized*. When the vibrations are elliptical, we speak of *elliptically polarized light*, and, in the special case that the ellipse becomes a circle, of *circularly polarized light*. According as the vibrations occur clockwise or anti-clockwise when viewed from the extremity of a vector whose direction is that of the ray, we say that circularly polarized light is *right-handed* or *left-handed*. Two linearly or plane polarized rays whose direction of propagation is the same, but whose vibrations occur at

right angles, are said to be *polarized at right angles to each other*.[3]

It follows from the considerations of § 39 that, in general, we can always *compound* two rays of light which are polarized at right angles to each other into an elliptically polarized ray, and conversely, we can *resolve* an elliptically polarized ray into two components of different phase vibrating at right angles to each other. Similarly, we can always resolve a plane polarized ray into two opposite circularly polarized rays, and conversely, we can combine two opposite circularly polarized rays of the same period, amplitude, and direction of propagation to form a plane polarized ray.

§ 78. Intensity and Pressure of Light.

The *intensity of a ray of light* is obviously determined by the magnitude of *Poynting's vector* which, as we know from § 68, measures the *flow of energy*. Hence [by § 68 eqn. (5)], the intensity of light is proportional to the product of the magnitudes of the electric and the magnetic strength of field. In any given medium, however, the magnetic field-strength is itself proportional to the electric field-strength [in accordance with § 57 eqn. (9)]; and therefore the intensity of light is proportional to the mean value of the square of the electric strength of field. We may thus say that the *intensity of light* is *determined by the square of the amplitude of the electric field-strength*.

If we consider spherical waves travelling away from a source of light, we know from § 43 eqn. (16) that the amplitude is inversely proportional to the distance from the centre. The *intensity of light* must therefore *decrease in inverse proportion to the square of the distance from the source of the light*.

According to § 68 eqn. (1), the *energy density of a light wave in vacuo* (or, what is practically the same, in air), is given by the simple relation

$$(1) \qquad \eta = \frac{E^2}{4\pi} = \frac{H^2}{4\pi};$$

for, by § 69 eqn. (7), E and H are equal to each other in free space. Both the quantity E^2 and the quantity H^2 vary,

---

[3] By this we mean that the electric vibrations in the one ray occur at right angles to those in the other ray, and similarly with the magnetic vibrations.

however, periodically ; and, denoting the amplitude by A, we have for the mean value of the square

(2) $$\overline{E^2} = A^2 \frac{\int_0^{2\pi} \sin^2 x\, dx}{2\pi}.$$

Let

(3) $$\int_0^{2\pi} \sin^2 x\, dx = I.$$

Then, as a graphical representation at once shows, we must also have

(4) $$\int_0^{2\pi} \cos^2 x\, dx = I.$$

Adding eqns. (3) and (4) we thus find

(5) $$\int_0^{2\pi} dx = 2I ;$$

and hence the *mean value of the energy density* (denoted by η) is given by

(6) $$\eta = \frac{A^2}{8\pi}.$$

From this expression we can calculate the amplitude of the electric and magnetic field-strengths, when the value of the energy density is known.

Let us take, as an example, the case of a *beam of sunlight* reaching the earth. As we know from exact measurements, the quantity of heat due to direct solar radiation that is received *in a minute* on one square centimetre, at the limits of the earth's atmosphere, is equal to 3·0 cal.[1] Since a calorie is equal to $4·2 \times 10^7$ ergs, a square centimetre absorbs *in a second* a quantity of energy equal to $2·1 \times 10^6$ ergs. This quantity of energy is contained within a prism whose base is 1 cm.² in area, and whose length is 3×10^{10} cm., for the energy travels with the velocity of light. The mean energy density per cubic centimetre is therefore

(7) $$\eta = 7 \times 10^{-5} \text{ erg/cm.}^3$$

Hence we obtain from eqn. (6) for the amplitude of the electric or the magnetic field-strength

(8) $$A = 0·04 \text{ absolute units.}$$

Since a volt is equal to the three-hundredth part of the absolute unit of electric potential, the amplitude of the

[1] At the earth's surface itself this quantity is about one-third smaller in amount, on account of the absorption of the radiation by the atmosphere.

electric strength of field in the beam of sunlight corresponds approximately to a potential gradient of 12 volts a centimetre. The electric field-strength is just as large as that produced by a charge of one electrostatic unit at a distance of about 5 cm. (The usual charges on Leyden jars amount to a few electrostatic units.) The amplitude of the magnetic strength of field in a beam of sunlight is found to be about one-fifth as large as the horizontal intensity of the earth's magnetic field.

It follows from § 69 that the pressure exerted by a light wave upon a body which completely absorbs it is equal in magnitude to the energy density : this is the so-called *pressure of light*. Thus, in the case of the beam of sunlight just considered, it amounts to 7×10^{-5} dynes a square centimetre ; or, since a dyne corresponds roughly to the weight of a milligram, to about three-quarters of a milligram a square metre.

The existence of the pressure of light was first *experimentally* demonstrated by *Lebedew* in 1901, by means of an exceedingly light mirror in a highly evacuated space. The experimental results quantitatively confirmed the accuracy of the relation, according to which the pressure of light is equal to the energy density.

§ 79. Interference of Light.

When *electric fields are superimposed*, the *strengths of field add up vectorially*, as is also the case with magnetic fields. Hence, from the electromagnetic nature of light follows at once the *optical principle of superposition* which had already been established in 1800 by *Young*, long before Maxwell's time. According to this, the resultant displacement in the light vibrations at a point where several rays meet simultaneously is found by addition of the individual displacements.[1]

Thus in particular, when two rays of light meet at a point, and have the same direction, the same amplitude, and the same period, but exhibit a *difference in path* (*cf.* § 42) equal to an *odd multiple of half a wave-length*, the two rays must annul each other. This phenomenon is known as the *interference of light*.[2] Under certain circumstances darkness

[1] Young, it is true, at first assumed the vibrating quantity to be a scalar, because at the time that he advanced the principle of superposition he still supposed light waves to be longitudinal.

[2] When the agreement in direction and amplitude is not complete, interference will naturally be only imperfect.

may thus result from the addition of light to light; and it is evident that the direct experimental confirmation of this phenomenon was destined finally to refute all theories which assume light to be composed of particles.

Such an experimental proof, however, was very difficult. If there is to be a noticeable interference between two rays of light, it is absolutely necessary that their *difference in phase* shall be *constant* throughout an interval of time sufficiently long to produce a definite impression of light. Rays satisfying this condition are called *coherent*. Now experience has shown that two rays arising from *different sources of light* are *never* coherent. *Fresnel* therefore demonstrated the interference of light by placing a source of natural light in front of two mirrors which were inclined at a very slight angle to each other. The two *images*, which in consequence of the small angle of inclination were very close together, then behaved as two different sources of light emitting, however, coherent rays (Fig. 54). By using *monochromatic* light, Fresnel was actually able to observe clearly black *interference bands* on a screen placed in front of the two mirrors. The bands

Fig. 54.

represent the geometrical positions of all those points on the screen whose distances from the two images differ by exact odd multiples of half a wave-length.

If, however, two similarly directed rays are *polarized at right angles to each other*, they can *never be made to interfere*. For, according to the Theorem of Pythagoras, the square of the resultant displacement is always equal to the sum of the squares of the two superimposed displacements, and hence the *resultant intensity* is always *equal to the sum* of the intensities of the two rays, so that they never weaken each other.

As a matter of fact, in 1816, *Fresnel* and *Arago* made the (to them) very surprising discovery that the two rays into which a ray of light is split by double refraction cannot under any circumstances be made to interfere, either completely or partially, although they are undoubtedly coherent. Indeed the total intensity is always found to be exactly equal to the sum of the intensities of the two separate rays.
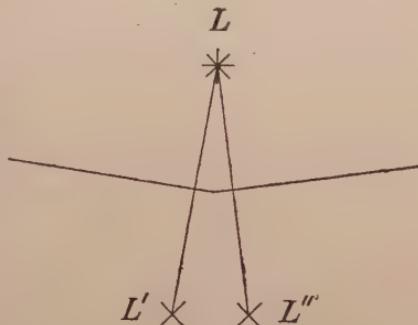
This experimental result of Fresnel and Arago led *Young* in 1817 to the conclusion that the hitherto generally accepted hypothesis of the longitudinal nature of light waves must be replaced by a conception of *transverse waves*. For only then did it appear possible for the vibrations to occur at right angles to each other in the two rays which could never be made to interfere.

By means of the principle of interference *Young* had already been able, in 1802, to evaluate the *wave-lengths of visible light*, on the basis of the so-called *Newton's rings*. These are observed when a slightly curved, thin glass lens is placed on a glass plate. Young quite rightly attributed these rings to the mutual interference of the rays of light reflected from the layer of air bounded by the lens and from the plane glass plate (Fig. 55). It is true that a difficulty was presented by the fact that, just where the lens touches the glass plate, there is a dark spot in the case of reflected monochromatic light. In order to explain this fact, Young had to have recourse to an assumption devised solely for this purpose, namely, that the amplitude is reversed on reflection at the glass plate; in other words, that *half a wave-length is thereby lost*. This assumption, which certainly must have appeared extremely artificial to Young's contemporaries, is actually (as we shall see later) a necessary consequence of Maxwell's equations.
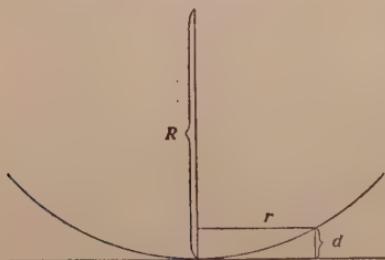
Fig. 55.

In the case of *normal incidence* and monochromatic light, darkness must thus appear at every place where the sum of twice the thickness (d) of the layer of air and of the half wave-length lost by reflection at the glass plate is equal to an odd multiple of a half wave-length. At such places we therefore have the relation

$$2d + \frac{\lambda}{2} = (2z + 1)\frac{\lambda}{2},$$

or

(1) $$2d = z\lambda,$$

where z denotes the series of integers. Let R be the radius

of curvature of the lens surface, and r the radius of a dark ring ; then

$$(2) \qquad r^2 = (2R - d)\, d\,,$$

for r represents the height of a right-angled triangle, the constituents of whose hypothenuse are equal to $(2R - d)$ and d. If d be small in comparison with the radius of curvature of the lens (which is only slightly curved), we have by eqn. (2)

$$(3) \qquad d = \frac{r^2}{2R}\,,$$

and hence

$$(4) \qquad \lambda = \frac{r^2}{zR}\,.$$

By measuring the radii of the dark rings, Young was thus able to calculate the *wave-length* of the light used, and likewise its *frequency*.

If the lens in Fig. 55 does not touch the plate, but is separated from it by a distance a, we have, in place of eqn. (1), the expression

$$(5) \qquad 2\,(d + a) = z\lambda\,;$$

where d now denotes, not the distance from the plate, but that from a plane parallel to the plate and at a distance a from it. We then find from eqns. (5) and (3)

$$(6) \qquad \frac{r^2}{R} = z\lambda - 2a\,.$$

Hence the further the lens is from the plate, the smaller is the radius of the ring whose ordinal number is z. The centre appears dark when $2a$ is equal to an even multiple of half a wave-length, but bright when $2a$ is equal to an odd multiple. Thus, if the lens be gradually drawn away from the plate (*e.g.*, by means of a fine screw), the rings will appear to contract towards the centre, where they will vanish. If the light is perfectly monochromatic, it must still be possible to observe the rings even for arbitrarily large values of a. Actually, however, they become continually fainter and eventually vanish completely, as the distance between lens and plate is increased. The greater the value of a at which this disappearance is initiated, the greater the *capacity for*

interference possessed by the light used; and this will obviously be greater, the more perfectly monochromatic the interfering light. Using the light of the finest spectral lines, interference has been observed up to a difference in path of 2,600,000 wave-lengths.[3]

§ 80.   The Laws of Reflection and Refraction.

In taking the *electromagnetic theory of light* as a basis for the investigation of the phenomena of *reflection* and *refraction* at the interface between two transparent media, we will begin by making the simplifying assumption that both media are perfect, homogeneous *insulators*, and that furthermore, the magnetic permeability of both is equal to unity (which, in fact, is very approximately true for most insulators). Let the dielectric constants of the two media be ϵ_1 and ϵ_2. We will choose the interface (which we may suppose plane) as the x-y-plane, and the plane determined by the direction of propagation of the incident ray and the incident normal as the x-z-plane. Let the z-axis be directed from the first to the second medium.

Suppose, now, a *plane polarized electromagnetic plane wave* to meet the interface at an *angle of incidence* φ;[1] let **A** be the amplitude of the electric strength of field and let τ be the period. We will make no further suppositions beyond assuming the incident ray to be *divided* into two, likewise plane rays, of which the one returns from the interface into the first medium, whilst the other advances into the second medium. The former ray we call the *reflected*, and the latter the *refracted* ray. Let the angles included between them and the incident normal be φ' and φ'', the angles between them and the incident or x-z-plane[2] be ω' and ω'', and their periods be τ' and τ''.

Let us now cut off any three lengths r, r', r'' in the directions of the three rays (or opposed to them, as the case may be) from the point where the ray under consideration meets the interface. Then, between these three segments and the coordinates of their extremities (which we shall denote by x, y, z; x', y', z'; x'', y'', z'') there exist the following

---

[3] Lummer and Gehrcke succeeded in demonstrating this with the aid of their so-called plate spectroscope, an instrument devised by them and based on similar principles.

[1] By the angle of incidence we mean, as is well known, the angle included between the direction of the incident ray and the normal to the interface.

[2] For we cannot assume these angles to be equal to zero as a matter of course. It must first be proved that such is the case.
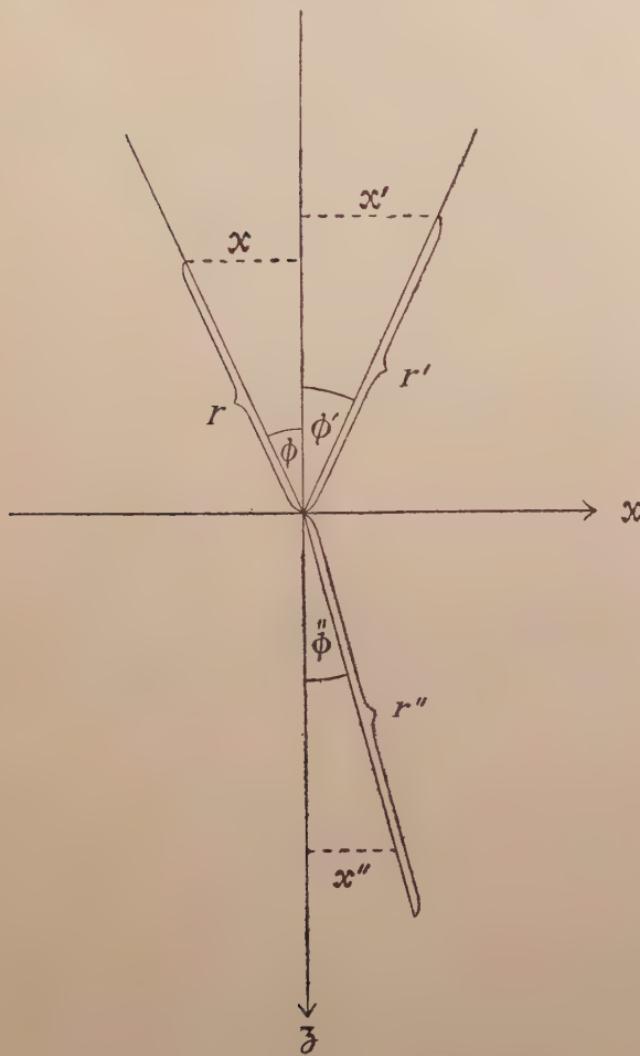
FIG. 56.

simple relations, as is at once obvious from a geometrical consideration of Fig. 56 :—

$$(1) \quad \begin{cases} x = -r \sin \varphi, & x' = r' \cos \omega' \sin \varphi', \\ & x'' = r'' \cos \omega'' \sin \varphi'', \\ y = 0, \ y' = r' \sin \omega', \ y'' = r'' \sin \omega'', \\ z = -r \cos \varphi, & z' = -r' \cos \omega' \cos \varphi', \\ & z'' = r'' \cos \omega'' \cos \varphi''. \end{cases}$$

From these we find that : [3]

$$(2) \quad \begin{cases} r = -x \sin \varphi - z \cos \varphi, \\ r' = x' \cos \omega' \sin \varphi' + y' \sin \omega' - z' \cos \omega' \cos \varphi', \\ r'' = x'' \cos \omega'' \sin \varphi'' + y'' \sin \omega'' + z'' \cos \omega'' \cos \varphi'' \end{cases}$$

Now, since by § 75 eqn. (4) $v = c/\sqrt{\epsilon_1}$, the *incident* plane polarized wave is determined by the equation

$$(3) \quad \mathbf{E} = \mathbf{A} \sin \left[\frac{2\pi}{\tau} \left(t + \frac{r}{c} \sqrt{\epsilon_1} \right) \right],$$

this vectorial formula corresponding to three scalar equations between the components of $\mathbf{E}$ and $\mathbf{A}$. The *reflected* ray is then described by the equation

$$(4) \quad E_x' = A_{(x)}' \sin \left[\frac{2\pi}{\tau'} \left(t - \frac{r'}{c} \sqrt{\epsilon_1} \right) + \delta_{(x)}' \right]$$

and by two analogous equations in which, however, we cannot forthwith assume either the quantities $A_{(x)}'$, $A_{(y)}'$, $A_{(z)}'$ to be the components of a vector, or the phase constants $\delta_{(x)}'$, $\delta_{(y)}'$, $\delta_{(z)}'$ to be identical. Similarly, the refracted ray is determined by the equation

$$(5) \quad E_x'' = A_{(x)}'' \sin \left[\frac{2\pi}{\tau''} \left(t - \frac{r''}{c} \sqrt{\epsilon_2} \right) + \delta_{(x)}'' \right]$$

together with two analogous equations. The values given by eqn. (2) for r, r', r'' are to be inserted in eqns. (3), (4) and (5).

We have learnt in § 73 that the *tangential component of the electric field-strength* must be *continuous in crossing the interface between two media*. Since the electric strength of field consists of $\mathbf{E}$ and $\mathbf{E}'$ in the first medium, while it is equal to $\mathbf{E}''$ in the second, we must have the following relations satisfied at that point in the interface where the reflection occurs :—

$$(6) \quad \begin{cases} E_x + E_x' = E_x'', \\ E_y + E_y' = E_y''. \end{cases}$$

The tangential components of the electric strength of field will naturally lie in the x-y-plane. But at this point, where

---

[3] *E.g.*, multiply the equation for x by $(-\sin \varphi)$ and that for z by $(-\cos \varphi)$ and then add,

the incident ray is split up, and which is thus common to all three rays, we obviously have

$$x = x' = x''\,; \quad y = y' = y''\,; \quad z = z' = z'' = 0\,.$$

Hence, bearing these relations and eqn. (2) in mind, and introducing into the first of the two eqns. (6) the values for E_x, E_x', and E_x'' as given by eqns. (3), (4), and (5), we find

$$
(7) \left\{
\begin{aligned}
&A_x \sin\left[\frac{2\pi}{\tau}\left(t - \frac{x \sin \varphi}{c}\sqrt{\epsilon_1}\right)\right] \\
&+ A_{(x)}' \sin\left[\frac{2\pi}{\tau'}\left(t - \frac{x \cos \omega' \sin \varphi' + y \sin \omega'}{c}\sqrt{\epsilon_1}\right) + \delta_{(x)}'\right] \\
&= A_{(x)}'' \sin\left[\frac{2\pi}{\tau''}\left(t - \frac{x \cos \omega'' \sin \varphi'' + y \sin \omega''}{c}\sqrt{\epsilon_2}\right) \right. \\
&\hspace{8cm} \left. + \delta_{(x)}''\right].
\end{aligned}
\right.
$$

Eqn. (7) and the analogous equation for the y-component of the electric field-strength must, however, be satisfied for *every arbitrary value of t* and, *independently thereof*, for every arbitrary value of x and, again independently thereof, for every arbitrary value of y. We can see that this is only possible if the following relations are satisfied :—

$$
(8) \quad \left\{
\begin{aligned}
A_x + A_{(x)}' &= A_{(x)}'', \\
A_y + A_{(y)}' &= A_{(y)}'';
\end{aligned}
\right.
$$

$$(9) \qquad \tau' = \tau'' = \tau\,;$$

$$(10) \qquad \delta_{(x)}' = \delta_{(x)}'' = \delta_{(y)}' = \delta_{(y)}'' = 0\,;$$

$$(11) \qquad \omega' = \omega'' = 0\,;$$

$$(12) \qquad \varphi' = \varphi\,;$$

$$(13) \qquad \frac{\sin \varphi}{\sin \varphi''} = \frac{\sqrt{\epsilon_2}}{\sqrt{\epsilon_1}}\,.$$

Eqn. (9) expresses the fact that the *periods* of the reflected and the refracted rays are *identical* with that of the incident ray, so that there is *no alteration in colour*. Eqn. (10) shows us that, in the case of reflection at an insulator, provided that there be also refraction, *no displacement in phase* whatever occurs in the components of the vibration, and hence that, when the incident ray is plane polarized, the reflected and the refracted rays will also be *plane polarized*. It follows

from eqn. (11) that the reflected and the refracted rays lie *in the plane of incidence*, determined by the incident ray and the normal to the interface. Eqn. (12) is an expression of the *law of reflection*, according to which the angle of reflection is equal to the angle of incidence.

In eqn. (13) we have Snellius' *law of refraction*, which states that the *ratio of the sines of the angles of incidence and refraction is independent of the angle of incidence*; it is completely determined by the *dielectric constants of the two media*. The ratio $\sin \varphi / \sin \varphi''$ is known as the *refractive index* of the second with respect to the first medium, and, in the special case in which a ray is refracted from free space into a medium, as the absolute refractive index (n) of the medium, or simply as its refractive index. From eqn. (13) follows the important expression, known as *Maxwell's relation*,

$$(14) \qquad\qquad \epsilon = n^2.$$

The *dielectric constant* of a medium is *equal to the square of its refractive index*.[4]

As *Boltzmann* showed in 1873, Maxwell's relation proves to be very well fulfilled in the case of *gases*, as appears from the following table (which refers to yellow light) :—

	n	$\sqrt{\epsilon}$
Air	1·000294	1·000295
Hydrogen . .	1·000138	1·000132
Carbon dioxide .	1·000449	1·000473
Carbon monoxide .	1·000346	1·000345
Nitrous oxide . .	1·000503	1·000497

Boltzmann's measurements on gases must thus be regarded as a striking *experimental confirmation of Maxwell's theory*. Apart from this, however, the Maxwell relation is not well fulfilled for waves of the frequency of visible light,[5] although this disagreement in no way controverts the electromagnetic

[4] It follows from eqn. (13) that the refractive index of one medium relative to another—in agreement with the well-known elementary law—is equal to the ratio of the refractive indices of the two media relative to free space (or, what is practically the same, to air).

[5] *E.g.*, in the case of water the dielectric constant is 81, but the refractive index is only 1·33 ; the dielectric constant for methyl alcohol is 32, whereas the refractive index is 1·34.

theory of light, for it finds an explanation in the *dispersion of the waves*. That such is the case is also proved by the fact that, in the case of *Hertzian waves* several metres in length, Maxwell's relation holds *exactly*.

Since the velocity of an electromagnetic wave is equal to $c/\sqrt{\epsilon}$ [by § 75 eqn. (4)],[6] Maxwell's relation also leads to a relation already well-known to the earlier theory of light, *viz.*, that the velocity of light in any given medium is equal to the velocity *in vacuo* divided by the refractive index of the medium.

§ 81. Fresnel's Equations.

The fact that the tangential component of the *electric* strength of field is continuous in crossing the interface between two insulators is expressed, in accordance with § 80 eqn. (8), by the two relations [1]

$$(1) \quad \begin{cases} A_x + A_x{}' = A_x{}'', \\ A_y + A_y{}' = A_y{}''. \end{cases}$$

In addition to these two equations, we have to take account of two analogous ones relating to the amplitudes of the *magnetic* waves. For, as a result of the complete analogy existing between magnetostatic and electrostatic phenomena (which are both based on the same Coulomb's Law), the vector of the magnetic strength of field will also, quite generally, be continuous in crossing the interface between two media. This will more readily be the case when, as we here assume, the value of the magnetic permeability is the same in both media—*viz.*, unity. But, according to § 75 eqn. (9), the magnetic field-strengths in the incident, the reflected, and the refracted waves are connected with the electric field-strengths by the relations

$$(2) \quad \mathbf{H} = \sqrt{\epsilon_1}\,[\mathbf{f}\,\mathbf{E}], \quad \mathbf{H}' = \sqrt{\epsilon_1}\,[\mathbf{f}'\,\mathbf{E}'], \quad \mathbf{H}'' = \sqrt{\epsilon_2}\,[\mathbf{f}''\,\mathbf{E}''],$$

where $\mathbf{f}$, $\mathbf{f}'$, $\mathbf{f}''$ are the unit-vectors lying in the directions of propagation of the three rays. Since, however, the amplitudes of the vibrating quantities $\mathbf{E}$, $\mathbf{E}'$, $\mathbf{E}''$ are equal to $\mathbf{A}$, $\mathbf{A}'$, $\mathbf{A}''$, the *continuity of the tangential components of the magnetic strength of field* leads, by analogy with eqn. (1),

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[6] μ has throughout been assumed to be unity.

[1] Since, in accordance with § 80 eqn. (10), the reflected and refracted rays are plane polarized when, as we have assumed, the incident ray is, we may regard $A(x)$, $A(y)$ as the components of a vector, and omit the brackets round the indices.

and taking Maxwell's relation into account, to the expressions

(3)
$$[\mathfrak{f}\mathbf{A}]_x + [\mathfrak{f}'\mathbf{A}']_x = n\,[\mathfrak{f}''\mathbf{A}'']_x,$$
$$[\mathfrak{f}\mathbf{A}]_y + [\mathfrak{f}'\mathbf{A}']_y = n\,[\mathfrak{f}''\mathbf{A}'']_y.$$

Here n denotes the refractive index in passing from the first medium to the second.

Undoubtedly our problem will now consist in *calculating*
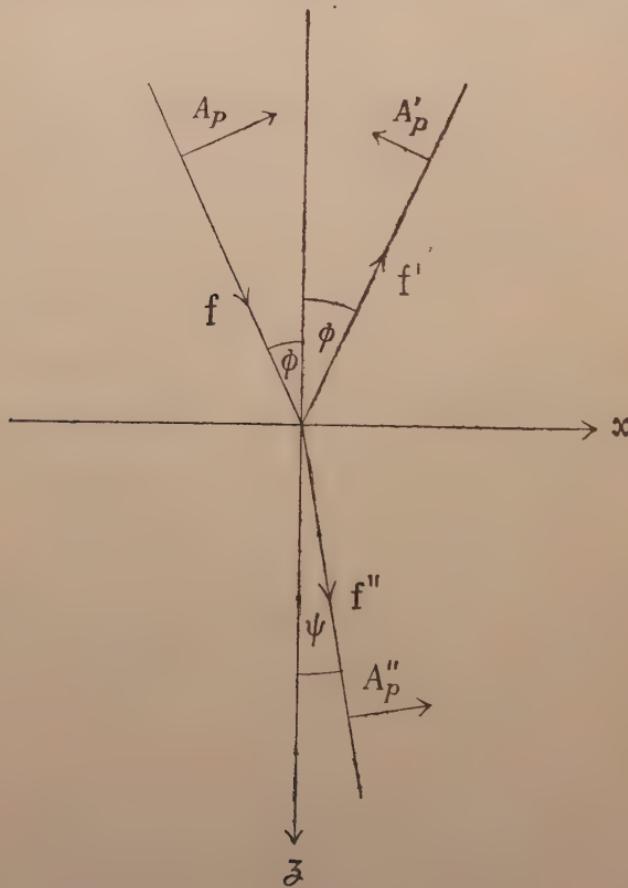
Fig. 57.

the two vector quantities **A'** *and* **A''**, for these *completely determine* the reflected and the refracted electromagnetic waves as regards both their *intensity* and their *direction of vibration*. Inasmuch as we already know two properties of the vectors **A'** and **A''**, *viz.*, that, in consequence of the transverse

nature of the vibrations of light, the vector $\mathbf{A}'$ must be perpendicular to the direction of $\mathbf{f}'$ as given by the law of reflection, and the vector $\mathbf{A}''$ perpendicular to the direction of $\mathbf{f}''$ as given by the law of refraction, it follows that the four eqns. (1) and (3) must actually suffice to determine the vectors $\mathbf{A}'$ and $\mathbf{A}''$ completely.

With this object in view, we will *resolve* each of the vectors $\mathbf{A}$, $\mathbf{A}'$, $\mathbf{A}''$ into two mutually perpendicular components which, like the vectors themselves, shall be at right angles to the direction of propagation of the ray. One set of the two components, which may be denoted by A_s, A_s', A_s'' respectively, shall be *normal to the incident plane*, their positive direction being that of the positive y-axis. The other components, which must therefore lie *in the incident plane*, and which may be denoted by A_p, A_p', A_p'' respectively, are to be considered positive when the rotation necessary to carry the component A_p (or A_p' or A_p'') into the direction of the component A_s (or A_s' or A_s'') appears anti-clockwise as seen from the extremity of the vector $\mathbf{f}$ (or $\mathbf{f}'$ or $\mathbf{f}''$). In Fig. 57 (in which, by virtue of the definition of an English coordinate system, the y-axis will be directed forwards from the plane of the figure) all three components A_p, A_p', A_p'' will thus be positive when drawn as shown.

Denoting the angle of refraction by ψ instead of by φ'' we find, by a geometrical consideration, the following simple relations :—

$$(4) \begin{cases} A_x = A_p \cos\varphi, & A_y = A_s, & A_z = -A_p \sin\varphi, \\ A_x' = -A_p' \cos\varphi, & A_y' = A_s', & A_z' = -A_p' \sin\varphi, \\ A_x'' = A_p'' \cos\psi, & A_y'' = A_s'', & A_z'' = -A_p'' \sin\psi. \end{cases}$$

We will now insert the values given by these equations in eqns. (3). Since, however, all three rays are propagated in the x-z-plane, the y-components of the unit-vectors $\mathbf{f}$, $\mathbf{f}'$, $\mathbf{f}''$ all vanish. Hence, in accordance with the rule for the vector product of two vectors, eqns. (3) become

$$(5) \begin{cases} f_z A_y + f_z' A_y' = n f_z'' A_y'', \\ f_z A_x - f_x A_z + f_z' A_x' - f_x' A_z' = n \left(f_z'' A_x'' - f_x'' A_z'' \right). \end{cases}$$

Now

$$(6) \begin{cases} f_x = \sin\varphi, & f_z = \cos\varphi, \\ f_x' = \sin\varphi, & f_z' = -\cos\varphi, \\ f_x'' = \sin\psi, & f_z'' = \cos\psi. \end{cases}$$

Introducing the values given by eqns. (4) and (6) into eqns.

(1) and (5), we obtain four equations which contain the *four unknowns* A_p', A_s', A_p'', A_s''. These equations are

(7)
$$\begin{cases} (A_p - A_p') \cos \varphi = A_p'' \cos \psi, \\ A_s + A_s' = A_s'', \\ (A_s - A_s') \cos \varphi = n A_s'' \cos \psi, \\ A_p + A_p' = n A_p''. \end{cases}$$

From these four equations we can find the four unknowns with the help of some simple calculations. We first find the value of A_p' by multiplying the first equation by n and the fourth by $\cos \psi$, and then equating the two left-hand sides. This gives

$$(A_p - A_p') n \cos \varphi = (A_p + A_p') \cos \psi.$$

From this we find

(8)
$$A_p' = A_p \frac{n \cos \varphi - \cos \psi}{n \cos \varphi + \cos \psi},$$

or, on putting

$$n = \frac{\sin \varphi}{\sin \psi}$$

in accordance with the law of refraction, and then multiplying the numerator and denominator of the fraction in eqn. (8) by $\sin \psi$,

(9)
$$A_p' = A_p \frac{\sin \varphi \cos \varphi - \sin \psi \cos \psi}{\sin \varphi \cos \varphi + \sin \psi \cos \psi}.$$

But

(10)
$$\begin{cases} \sin \varphi \cos \varphi \pm \sin \psi \cos \psi = \frac{1}{2} \{ \sin 2\varphi \pm \sin 2\psi \} \\ = \sin (\varphi \pm \psi) \cos (\varphi \mp \psi). \end{cases}$$

Hence

(11)
$$A_p' = A_p \frac{\tan (\varphi - \psi)}{\tan (\varphi + \psi)}.$$

In like manner we can calculate A_s' from the second and third of eqns. (7) by multiplying the second equation by $n \cos \psi$. We then have

$$(A_s + A_s') n \cos \psi = (A_s - A_s') \cos \varphi,$$

whence

(12)
$$A_s' = - A_s \frac{n \cos \psi - \cos \varphi}{n \cos \psi + \cos \varphi},$$

or, again replacing n by the quotient $\sin \varphi / \sin \psi$ and multiplying numerator and denominator by $\sin \psi$,

(13) $$A_s' = - A_s \frac{\sin (\varphi - \psi)}{\sin (\varphi + \psi)}.$$

On taking eqn. (9) into account, we find from the fourth of eqns. (7)

$$A_p'' = A_p \left[\frac{\sin \psi}{\sin \varphi} + \frac{(\sin \varphi \cos \varphi - \sin \psi \cos \psi) \sin \psi}{(\sin \varphi \cos \varphi + \sin \psi \cos \psi) \sin \varphi} \right].$$

Bringing both fractions to the same denominator, we obtain

$$A_p'' = A_p \frac{2 \sin \varphi \cos \varphi \sin \psi}{\sin \varphi (\sin \varphi \cos \varphi + \sin \psi \cos \psi)};$$

or, bearing in mind eqn. (10),

(14) $$A_p'' = A_p \frac{2 \cos \varphi \sin \psi}{\sin (\varphi + \psi) \cos (\varphi - \psi)}.$$

Finally, the second of eqns. (7) gives us, when eqn. (13) is taken into account,

$$A_s'' = A_s \left[1 - \frac{\sin (\varphi - \psi)}{\sin (\varphi + \psi)} \right],$$

or,

(15) $$A_s'' = A_s \frac{2 \cos \varphi \sin \psi}{\sin (\varphi + \psi)}.$$

The four equations (11), (13), (14), (15) are known as *Fresnel's equations*, after the investigator who first derived them (in 1821). Fresnel, however, obtained them on the basis of the elastic theory of light, with the help of some *ad hoc* hypotheses.

§ 82. Polarization by Reflection and by Refraction.

Fresnel's equations determine both the *intensity* and the *direction of vibration* in the reflected and the refracted ray, and hence we are also in a position to derive all *changes in the direction of vibration* produced by reflection or refraction. The first of Fresnel's equations [§ 81 eqn. (11)] shows that A_p' must always *vanish* when

(1) $$\varphi + \psi = \frac{\pi}{2}.$$

This will be the case for that value of the angle of incidence for which

$$\sin \psi = \cos \varphi ,$$

or, according to the law of refraction,

(2)　　　　　　　　$$\frac{1}{n} \sin \varphi = \cos \varphi .$$

This definite angle (φ^*) is therefore determined by the relation

(3)　　　　　　　　$$\tan \varphi^* = n.$$

When light with any arbitrary plane of vibration meets a reflecting surface at this angle, the vibrations of the *reflected* electric wave will be exclusively normal to the incident plane, for A_p' will vanish, but not A_s'. Hence a ray of *natural* light will give rise to *plane polarized* light on reflection at this angle. The angle φ^* is consequently called the *angle of polarization*. The fact that its trigonometrical tangent is equal to the refractive index [1] is known as *Brewster's law*, after the investigator who first (in 1815) discovered this relation.

Malus, who discovered polarization by reflection in 1808, defined the incident plane as the *plane of polarization* of the reflected ray. It thus follows from this arbitrary definition that the *electric vibrations* must take place *at right angles to the plane of polarization*, while the *magnetic vibrations*, which are normal to the electric, must occur *in the plane of polarization*.[2]

When, in particular, a ray of light which is already polarized strikes the reflecting surface at the angle of polarization, and when the vibrations occur in the incident plane so that A_s is equal to zero, it follows from § 81 eqn. (13) that A_s', as well as A_p', vanishes. In that case there is consequently *no reflection* at all. In such a way it was possible for *Malus* to produce an apparent extinction of a ray of light by means of two successive reflections at the angle of polarization from two crossed mirrors.

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[1] *E.g.*, in the case of glass the angle of polarization is 56°.

[2] In the elastic theory of light *Fresnel's* fundamental assumption that the elasticity of the æther was the same in different media, but that its density differed, led to the conclusion that the vibrations of light must be *at right angle to the plane of polarization*. On the other hand, on the assumption that the density of the æther was the same, but that its elasticity differed in different media, *Franz Neumann* came to the conclusion that the vibrations of light must take place *in the plane of polarization*. In terms of the electromagnetic theory of light, therefore, the electric vibrations correspond to the light vibrations in the sense of Fresnel's theory, and the magnetic vibrations to the light vibrations in the sense of Neumann's theory.

Finally, when a ray of light which is already plane polarized, but whose direction of vibration is quite arbitrary, meets a reflecting surface, we find, on dividing the second of Fresnel's equations by the first,

(4)
$$\begin{cases} \dfrac{A_s{}'}{A_p{}'} = - \dfrac{A_s}{A_p} \cdot \dfrac{\cos (\varphi - \psi)}{\cos (\varphi + \psi)} \\[2mm] \qquad = - \dfrac{A_s}{A_p} \cdot \dfrac{\cos \varphi \cos \psi + \sin \varphi \sin \psi}{\cos \varphi \cos \psi - \sin \varphi \sin \psi} \cdot \end{cases}$$

Since both φ and ψ are always less than $\pi/2$, and hence the sine and cosine of both angles always positive, it follows that, in absolute magnitude,

(5)
$$\left| \frac{A_s{}'}{A_p{}'} \right| \; > \; \left| \frac{A_s}{A_p} \right|.$$

The vector **A'** therefore makes a larger angle with the incident plane than the vector **A**, *i.e.*, the *plane of polarization* of a ray of light is invariably *rotated towards the incident plane on reflection.*[3]

On the other hand, by dividing the fourth of Fresnel's equations [§ 81 eqn. (15)] by the third [§ 81 eqn. (14)], we find

(6)
$$\frac{A_s{}''}{A_p{}''} = \frac{A_s}{A_p} \cos (\varphi - \psi).$$

Hence, in every case,

(7)
$$\frac{A_s{}''}{A_p{}''} \; < \; \frac{A_s}{A_p} \, ;$$

i.e., the *plane of polarization* of the *refracted ray* is always *rotated away from the incident plane.*

We see from § 81 eqn. (15), however, that the component $A_s{}''$ can never vanish, unless A_s was originally zero, so that simple refraction can never produce a polarized ray from a ray of natural light. By transmitting a natural ray of light through a *series of glass plates*, however, we can produce a ray which behaves in effect like a completely polarized ray on the far side of the series, owing to the successive rotations of the plane of polarization.

Fresnel's equations assume especially simple forms when the angle of incidence (and, consequently, also the angle of refraction) is zero. It is true that in this special case of so-called *normal incidence* the equations cannot be applied

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[3] If the plane of polarization actually coincides with the incident plane then, of course, $A_s{}'/A_p{}'$ becomes infinitely large.

in their final form, because both numerator and denominator in the fractions vanish. We can avoid this difficulty, however, by using the equations in their unfinished form. The first equation [which we shall apply as given in § 81 eqn. (8)] becomes, in the case of normal incidence,

$$(8) \qquad A_p' = A_p \frac{n-1}{n+1}.$$

The second [§ 81 eqn. (12)] becomes

$$(9) \qquad A_s' = -A_s \frac{n-1}{n+1}.$$

In place of the third, we find from § 81 eqns. (7)

$$A_p'' = \frac{1}{n}(A_p + A_p'),$$

or,

$$(10) \qquad A_p'' = A_p \frac{2}{n+1}.$$

For the fourth we obtain, likewise from § 81 eqns. (7),

$$A_s'' = A_s + A_s',$$

or,

$$(11) \qquad A_s'' = A_s \frac{2}{n+1}.$$

In the case of normal incidence, the ratio of the intensity of the reflected ray to that of the incident ray is known as the *reflective power* of the substance whose refractive index is n. Since the intensity is determined by the square of the amplitude, we find for the reflective power

$$\varrho = \frac{A_p'^2 + A_s'^2}{A_p^2 + A_s^2},$$

or, by eqns. (8) and (9),

$$(12) \qquad \varrho = \left(\frac{n-1}{n+1}\right)^2.$$

It follows from this formula that in the case of glass, whose refractive index is about 1·5, the reflective power is approximately 1/25.

Eqn. (9) shows us that the component A_s' must always be opposite in sign to the component A_s when $n > 1$, *i.e.*, when the reflection takes place, as we say, at the optically denser medium. Since, however, the positive directions of both components A_s and A_s' are by definition to be taken as alike (*viz.*, in accordance with § 81, in the direction of the positive

y-axis), the opposite signs show that their directions must also be opposite. On the other hand, it follows from eqn. (8) that the components A_p and $A_p{}'$ have the same sign on reflection at an optically denser medium. But we see from Fig. 57, in which (as was mentioned at the time) the quantities A_p, $A_p{}'$, $A_p{}''$ are all positive as drawn, that, in the case of normal incidence, A_p and $A_p{}'$ are oppositely directed although identical in sign.

Hence, *on reflection at an optically denser medium*, the vector $\mathbf{A}'$, whose components are $A_s{}'$ and $A_p{}'$, must always be *opposite in direction* to the vector $\mathbf{A}$, whose components are A_s and A_p. This, however, simply means that the *amplitude* of the electric wave is *reversed* on reflection at an optically denser medium, *i.e.*, that the *electric wave* undergoes a *displacement in phase* of a *half-period*. An assumption which Young had to make expressly in order to explain Newton's rings thus results, in the electromagnetic theory of light, from a purely mathematical consideration. Consequently, immediately at the interface, $\mathbf{E}'$ and $\mathbf{E}$ are opposite in direction. Now by § 75 eqn. (9),[4]

$$(13) \qquad \mathbf{H} = \sqrt{\epsilon_1}\,[\mathfrak{f}\mathbf{E}]\,, \qquad \mathbf{H}' = \sqrt{\epsilon_1}\,[\mathfrak{f}'\mathbf{E}']\,.$$

Since, however, not only $\mathbf{E}$ and $\mathbf{E}'$ are oppositely directed but also, in the case of normal incidence, $\mathfrak{f}$ and $\mathfrak{f}'$ (the unit-vectors lying in the directions of propagation of the incident and the reflected ray), $\mathbf{H}$ and $\mathbf{H}'$ will be *similarly directed* at the reflecting surface, provided $n > 1$. Hence, whilst the electric wave experiences a displacement in phase on reflection at an optically denser medium, the magnetic wave does *not*.

It is easy to see that the reverse will be the case when the reflection takes place at an *optically rarer* medium. We then have $A_s{}'$ and A_s alike in sign, but $A_p{}'$ and A_p opposite. It follows from what we have said above that $A_s{}'$ and A_s will be similarly directed, but so also will $A_p{}'$ and A_p. On reflection at an optically rarer medium the *electric wave* therefore undergoes *no displacement*, whilst, in accordance with eqn. (13) (and because $\mathfrak{f}$ and $\mathfrak{f}'$ are opposite in direction), the *magnetic amplitude is reversed.*

Now, by interference of the incident and the reflected wave, *stationary electromagnetic waves* may theoretically be produced. If the reflection takes place at the optically denser medium, it follows from what we have just said that a *node* of vibration of the *electric* field-strength will arise at

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[4] μ has been put equal to unity.

the reflecting surface itself (seeing that the incident and the reflected electric waves differ in phase by a half-period). On the other hand, an *antinode* in the *magnetic* strength of field will arise at the interface, because, owing to the directions of vibration of the incident and the reflected ray being the same, the magnetic strengths of field will be simply added together. It is true that perfect nodes and antinodes can only be formed when the intensity of the reflected ray is approximately the same as that of the incident; otherwise only minima and maxima of intensity can occur. If reflection occurs at the optically rarer medium, there will be an antinode in the electric field-strength at the reflecting surface, and a node in the magnetic field-strength. In every case, however, the *electric nodes coincide with the magnetic antinodes* in the stationary wave, and *vice versâ*. Hence, in a stationary electromagnetic wave, the electric and the magnetic wave appear *displaced* with respect to each other, the displacement amounting to a *quarter-period* of the incident wave.

After *Hertz* had already demonstrated the production of stationary vibrations in the waves he himself discovered, on reflection at a metallic surface, *Otto Wiener* succeeded (in 1889) in demonstrating *stationary waves of light* by *photographic* means. Wiener placed a glass plate, covered with a film of silver chloride in collodion, in front of a metallic mirror and at a slight angle to it; he then illuminated the mirror with perpendicular rays from a monochromatic source. The film was so thin that its thickness only amounted to about 1/30th of a wave-length. Wiener was then actually able to demonstrate the existence of nodes and antinodes by the photographic action on the layer of silver chloride. Since this action did not appear at the points where nodes were theoretically to be expected in the *electric* wave, Wiener's experiments lead to the conclusion that it is the vibrations of the *electric* strength of field, and not of the magnetic strength of field at right angles thereto, which are the cause of the actual effects produced by light.

§ 83. Total Reflection.

We see from the equation which expresses the *law of refraction*,

(1) $$\sin \psi = \frac{1}{n} \sin \varphi,$$

that, when $n < 1$ and the refraction thus takes place from

the optically denser into the optically rarer medium, for a certain value of the angle of incidence the sine of ψ becomes unity. For still larger values of ψ, $\sin \psi$ becomes greater than unity, and the *angle of refraction* is then *imaginary*. In that case there is no refraction at all, as confirmed by experiment; instead, we have so-called *total reflection*.[1]

It is, however, quite possible to apply *Fresnel's equations* to the case of total reflection—a proceeding which is justified by the agreement of the results so obtained with experimental facts—provided that the quantity $\sin \psi$ be now taken to mean, not the sine of a definite angle, but merely the quotient $(\sin \varphi) / n$. We then have

$$(2) \qquad \cos \psi = \sqrt{1 - \frac{\sin^2 \varphi}{n^2}} = \frac{i}{n} \sqrt{\sin^2 \varphi - n^2}.$$

Fresnel's first equation [which we shall use in the preliminary form of § 81 eqn. (9)] now becomes

$$A_p' = A_p . \frac{\sin \varphi \cos \varphi - \dfrac{1}{n} \sin \varphi . \dfrac{i}{n} \sqrt{\sin^2 \varphi - n^2}}{\sin \varphi \cos \varphi + \dfrac{1}{n} \sin \varphi . \dfrac{i}{n} \sqrt{\sin^2 \varphi - n^2}} .$$

Dividing numerator and denominator by $\sin \varphi$, we may write this briefly

$$(3) \qquad A_p' = A_p \frac{a - ib}{a + ib},$$

where

$$(4) \qquad a = \cos \varphi, \qquad b = \frac{1}{n^2} \sqrt{\sin^2 \varphi - n^2},$$

a and b being always real quantities.

Separating the real terms from the imaginary in eqn. (3), we may write it in the form [2]

$$(5) \qquad A_p' = A_p \frac{a^2 - b^2}{a^2 + b^2} - 2iA_p \frac{ab}{a^2 + b^2} .$$

Now, as is well known, any arbitrary *complex* quantity $(G + iH)$ may be represented in the form

$$G + iH = \sqrt{G^2 + H^2} (\cos \alpha + i \sin \alpha),$$

[1] The angle of incidence for which $\sin \psi$ is unity is called the critical angle; *e.g.*, in the case of glass (with respect to air) it is 42°. Its observation renders possible an accurate determination of the refractive index.

[2] We do this by multiplying numerator and denominator of the fraction in eqn. (3) by the conjugate value of the denominator, *i.e.*, by $(a - ib)$.

where

$$\alpha = \tan^{-1}\frac{H}{G} . \text{[3]}$$

In accordance with *de Moivre's formula* [§ 38 eqn. (12)] this may also be written

(6) $$G + iH = \sqrt{G^2 + H^2} . e^{i\alpha} .$$

Using this formula, we will transform the expression for A_p' in eqn. (5).   Now, since

$$\left(\frac{a^2 - b^2}{a^2 + b^2}\right)^2 + \left(\frac{2 ab}{a^2 + b^2}\right)^2 = 1 ,$$

the absolute magnitude of the complex quantity A_p' is found to be equal to the real quantity A_p.   Hence we may write eqn. (5) in the form

(7) $$A_p' = A_p . e^{i\delta_p} ,$$

where, as a comparison of eqn. (7) with eqn. (3) shows,

(8) $$e^{i\delta_p} = \frac{a - ib}{a + ib} .$$

Now it follows from § 42 eqn. (10) that, quite generally, the *state of vibration* in a plane wave may be *symbolically* represented by the equation

(9) $$S = A . e^{2\pi i\left(\frac{t}{\tau} - \frac{r}{\lambda}\right)} ,$$

where either *only* the real or *only* the imaginary term (omitting the factor i) in the *complex* expression on the right-hand side of this equation is to be taken.[4]   In accordance with eqn. (9), the state of vibration in the reflected ray may, quite generally, be symbolically represented by the equation

(10) $$E_p' = A_p' . e^{2\pi i\left(\frac{t}{\tau} - \frac{r'}{\lambda}\right)} ,$$

both when reflection is accompanied by refraction and in the case of total reflection.

When, in particular, we are dealing with total reflection, we may combine eqns. (10) and (7), and write

(11) $$E_p' = A_p . e^{i\left[2\pi\left(\frac{t}{\tau} - \frac{r'}{\lambda}\right) + \delta_p\right]} .$$

On resolving this expression into its real and imaginary

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[3] The angle α is called the argument, and the square root $\sqrt{G^2 + H^2}$ the absolute magnitude of the complex quantity $(G + iH)$.

[4] S will be represented as a cosine or a sine function according as we use the real or the imaginary terms.

terms, *i.e.*, into a cosine and a sine function, we recognize that the component of the totally reflected electric wave which vibrates in the incident plane appears *displaced in phase* by an amount δ_p with respect to the incident wave.

In the case of the component of $\mathbf{A}'$ normal to the incident plane, *i.e.*, of A_s', we find from Fresnel's second equation [§ 81 eqn. (13)]

$$A_s' = -A_s \frac{\sin \varphi \cos \psi - \cos \varphi \sin \psi}{\sin \varphi \cos \psi + \cos \varphi \sin \psi},$$

or [by eqns. (1) and (2)],

$$A_s' = A_s \frac{\cos \varphi - i\sqrt{\sin^2 \varphi - n^2}}{\cos \varphi + i\sqrt{\sin^2 \varphi - n^2}}.$$

Introducing the abbreviations given by eqn. (4), we have

(12)
$$A_s' = A_s \frac{a - in^2 b}{a + in^2 b}.$$

By exactly the same train of thought as led us from eqn. (3) to eqn. (7), but substituting the quantity $n^2 b$ for b, we find that A_s' may be represented in the form

(13)
$$A_s' = A_s \cdot e^{i\delta_s}$$

where, as a comparison of eqn. (13) with eqn. (12) shows,

(14)
$$e^{i\delta_s} = \frac{a - in^2 b}{a + in^2 b}.$$

On dividing eqn. (8) by eqn. (14) and denoting the difference $(\delta_p - \delta_s)$ by δ, we find

$$e^{i\delta} = \frac{a - ib}{a + ib} \cdot \frac{a + in^2 b}{a - in^2 b}.$$

Giving a and b their values from eqn. (4), and carrying out the multiplications indicated, we obtain

$$e^{i\delta} = \frac{n^2 \cos^2 \varphi - i \cos \varphi (1 - n^2) \sqrt{\sin^2 \varphi - n^2} + \sin^2 \varphi - n^2}{n^2 \cos^2 \varphi + i \cos \varphi (1 - n^2) \sqrt{\sin^2 \varphi - n^2} + \sin^2 \varphi - n^2},$$

or, on dividing numerator and denominator by $(1 - n^2)$,

(15)
$$e^{i\delta} = \frac{\sin^2 \varphi - i \cos \varphi \sqrt{\sin^2 \varphi - n^2}}{\sin^2 \varphi + i \cos \varphi \sqrt{\sin^2 \varphi - n^2}}.$$

We shall write this equation briefly in the form

(16)
$$e^{i\delta} = \frac{g - ih}{g + ih}.$$

In order to calculate the angle δ, we will make use

of a relation which can be obtained from de Moivre's formula :

(17) $$e^{i\delta} + e^{-i\delta} = 2\cos\delta\,.$$

Now,

(18) $$e^{-i\delta} = \frac{g + ih}{g - ih},$$

and hence we obtain, by adding together eqns. (16) and (18),

(19) $$2\cos\delta = 2\,\frac{g^2 - h^2}{g^2 + h^2}\,.$$

This equation, however, is not particularly suitable for the calculation of δ, and we shall therefore derive $\delta/2$ in accordance with the well-known trigonometrical formula

(20) $$\tan\frac{\delta}{2} = \sqrt{\frac{1 - \cos\delta}{1 + \cos\delta}}$$

(omitting the sign from before the root as having no significance for us). It follows from eqn. (19) that

$$1 - \cos\delta = \frac{2h^2}{g^2 + h^2}\,,$$

$$1 + \cos\delta = \frac{2g^2}{g^2 + h^2}\,,$$

and hence that

(21) $$\tan\frac{\delta}{2} = \frac{h}{g}\,.$$

Substituting for h and g the values found by comparing eqns. (15) and (16), we finally obtain

(22) $$\tan\frac{\delta}{2} = \frac{\cos\varphi\,\sqrt{\sin^2\varphi - n^2}}{\sin^2\varphi}\,.$$

We see from this formula that, in the case of so-called *grazing incidence*, viz., when $\varphi = \pi/2$, δ vanishes. Otherwise, however, δ will always differ from zero, so that, in general, *totally reflected light will be elliptically polarized* (in consequence of the difference in phase between the two mutually perpendicular components). When the incident ray is already elliptically polarized, E_p and E_s differing in phase by δ_0, the components of the totally reflected electric wave will differ in phase by $(\delta_0 + \delta)$, where δ has the value given by eqn. (22). Hence, by a suitable choice of the angle of incidence, we can transform an elliptically polarized ray of light into a plane polarized ray.

It follows from eqn. (11) that the components of the electric amplitude lying in the incident plane are equally large in the reflected and the incident light, and analogously from eqn. (13) for those which are normal to the incident plane. Now, if A_s and A_p are equal in magnitude, which is the case when the incident ray is plane polarized and its plane of polarization makes an angle of 45° with the incident plane, the two mutually perpendicular components of the electric amplitude in the reflected light must also be equally large. If, therefore, a ray which is " plane polarized, with an azimuth of 45° with respect to the incident plane " strikes the reflecting surface at such an angle φ that δ in eqn. (22) is equal to $\pi/2$, the totally reflected ray must be *circularly polarized*. A simple calculation (upon which we shall not enter here) shows, however, that the necessary refractive index is so large that it is only in the case of *diamond* that plane polarized light can be changed into circularly polarized light by *a single* total reflection. But, in accordance with what we have said above, we can also produce circularly polarized light by *two successive total reflections*, each of which results in the *phases* of the two mutually perpendicular components being *displaced by one-eighth of a period* relatively to each other (without altering the magnitude of their amplitudes). We can

FIG. 58.

calculate from eqn. (22) that such will be the case for glass, with a refractive index of 1·5, both for an angle of incidence of 48·5° and for one of 54·5°.

The principle we have just mentioned forms the basis of *Fresnel's rhomb*, which is represented in Fig. 58, and which affords an excellent method for producing circularly polarized light. The rhomb is cut with an angle α corresponding to a displacement in phase of one-eighth of a period. The ray of light, which is plane polarized at an angle of 45° with respect to the plane of the figure, enters the rhomb normally at A, is totally reflected at B and C, and leaves the rhomb again normally at D, being then circularly polarized.
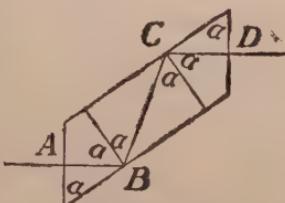
§ 84. The Optical Behaviour of Metals.

Our optical considerations have hitherto been limited to insulators, but we will now investigate the optical behaviour of *metals, i.e.,* of *conductors of electricity.* Here we shall

denote the *specific conductivity* by σ,[1] and § 74 eqns. (5) and
(6), of which we shall make use, then become

$$
\text{(1)} \quad
\begin{cases}
\operatorname{rot} \mathbf{H} = \dfrac{4\pi}{c} \sigma \mathbf{E} + \dfrac{\epsilon}{c} \dfrac{\partial \mathbf{E}}{\partial t}, \\[2ex]
\operatorname{rot} \mathbf{E} = -\dfrac{\mu}{c} \dfrac{\partial \mathbf{H}}{\partial t}.
\end{cases}
$$

We begin by forming the expressions rot rot $\mathbf{H}$ and rot rot $\mathbf{E}$
in accordance with a well-known rule of vectorial algebra
[§ 32 eqn. (25)]. Assuming that the medium contains no
real electricity, and that it is, moreover, homogeneous (so
that ϵ and μ are independent of position), we have

$$
\text{(2)} \qquad \operatorname{div} \mathbf{H} = \operatorname{div} \mathbf{E} = 0.
$$

Hence

$$
\text{(3)} \quad
\begin{cases}
\operatorname{rot} \operatorname{rot} \mathbf{H} = -\nabla^2 \mathbf{H}, \\
\operatorname{rot} \operatorname{rot} \mathbf{E} = -\nabla^2 \mathbf{E}.
\end{cases}
$$

Taking eqns. (1) into account we therefore find

$$
\text{(4)} \quad
\begin{cases}
\nabla^2 \mathbf{H} = -\dfrac{4\pi\sigma}{c} \operatorname{rot} \mathbf{E} - \dfrac{\epsilon}{c} \dfrac{\partial}{\partial t}(\operatorname{rot} \mathbf{E}), \\[2ex]
\nabla^2 \mathbf{E} = \dfrac{\mu}{c} \dfrac{\partial}{\partial t}(\operatorname{rot} \mathbf{H}),
\end{cases}
$$

or, when rot $\mathbf{E}$ and rot $\mathbf{H}$ are given their values from
eqns. (1),

$$
\text{(5)} \qquad \nabla^2 \mathbf{E} = \frac{4\pi\mu\sigma}{c^2} \frac{\partial \mathbf{E}}{\partial t} + \frac{\epsilon\mu}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2}
$$

and

$$
\text{(6)} \qquad \nabla^2 \mathbf{H} = \frac{4\pi\mu\sigma}{c^2} \frac{\partial \mathbf{H}}{\partial t} + \frac{\epsilon\mu}{c^2} \frac{\partial^2 \mathbf{H}}{\partial t^2}.
$$

For the moment we shall only consider eqn. (5), seeing
that eqn. (6) is completely analogous to it. We know from
our considerations of § 75 that, in the special case for which
σ is equal to zero, it has a particular solution in the form of
an expression which describes a plane, absorption-free,
electro-magnetic wave. If we further make the x-axis
coincide with the direction of propagation, so that simply

---

[1] Both the symbols λ and σ are used to denote specific conductivity, although
λ is also generally used for wave-length, and σ for surface-density of electricity.
We have, consequently, denoted specific conductivity in the theory of electricity
by λ, while in optics we may conveniently use the symbol σ.

$\partial^2\mathbf{E}/\partial x^2$ may be written for $\nabla^2\mathbf{E}$, the particular solution takes the form

(7)
$$\mathbf{A} \cdot e^{\frac{2\pi i}{\tau}(t - \kappa z)},$$

where κ is equal to $\sqrt{\epsilon\mu}/c$. (κ therefore stands for the reciprocal of the quantity which it denoted in § 43).

The idea will at once occur to us to try whether this expression will also satisfy eqn. (5), in which σ is not equal to zero, although the constant κ will most likely then have another signification than that maintaining when σ vanishes. If, *by way of trial*, we put $\mathbf{E}$ equal to the expression (7), replacing κ by κ', we find (since, in accordance with what we have said above, we may write simply $\partial^2\mathbf{E}/\partial x^2$ for $\nabla^2\mathbf{E}$)

(8)
$$\begin{cases} \nabla^2\mathbf{E} = -\dfrac{4\pi^2\kappa'^2}{\tau^2}\mathbf{E}, \\[2mm] \dfrac{\partial\mathbf{E}}{\partial t} = \dfrac{2\pi i}{\tau}\mathbf{E}, \\[2mm] \dfrac{\partial^2\mathbf{E}}{\partial t^2} = -\dfrac{4\pi^2}{\tau^2}\mathbf{E}. \end{cases}$$

On inserting the expressions (8) in eqn. (5), we obtain the relation

$$\frac{4\pi^2\kappa'^2}{\tau^2} = -\frac{8\pi^2\mu\sigma i}{c^2\tau} + \frac{4\pi^2\epsilon\mu}{c^2\tau^2},$$

or,

(9)
$$\kappa'^2 = \frac{\epsilon\mu}{c^2} - i\cdot\frac{2\mu\sigma\tau}{c^2}.$$

We thus see that the expression (7) for $\mathbf{E}$ does actually satisfy eqn. (5), provided that κ' be regarded as *complex*. We can put

(10)
$$\kappa' = \frac{\alpha - i\beta}{c},$$

where

(10a)
$$\alpha^2 - \beta^2 = \epsilon\mu$$

and

(10b)
$$\alpha\beta = \mu\sigma\tau.$$

From this we find

(11)
$$\begin{cases} \alpha = \sqrt{\dfrac{\mu}{2}\left(\sqrt{\epsilon^2 + 4\sigma^2\tau^2} + \epsilon\right)}, \\[3mm] \beta = \sqrt{\dfrac{\mu}{2}\left(\sqrt{\epsilon^2 + 4\sigma^2\tau^2} - \epsilon\right)}. \end{cases}$$

Inserting the value given by eqn. (10) in the expression (7), we thus obtain a particular solution of eqn. (5) in the symbolical form

$$\mathbf{E} = \mathbf{A} \cdot e^{\frac{2\pi i}{\tau}\left(t - \frac{xa}{c} + \frac{ix\beta}{c}\right)} ,$$

or (since $ii = -1$)

(12) $$\mathbf{E} = \mathbf{A} \cdot e^{-\frac{2\pi x\beta}{c\tau}} \cdot e^{\frac{2\pi i}{\tau}\left(t - \frac{xa}{c}\right)} .$$

This solution [2] of the differential equation (5) shows us that the electric wave is *absorbed in the conductor*; and, on account of the complete analogy between eqns. (5) and (6), exactly the same will naturally hold good for the magnetic wave. Maxwell's theory therefore leads to the conclusion (which is confirmed by experience) that all *conductors of electricity* must be *opaque,* provided their thickness be not too small.

It is true that relations are established by eqns. (11) between the values of α and β, which represent the *optical constants of the metal*, and the dielectric constant ϵ, but the value of the latter is unknown in the case of metals ; conversely, it cannot be determined by observations, for instance, of β. Since the value of σ for metals is of the order 10^{17} in absolute electrostatic measure, the unknown quantity ϵ can be neglected in comparison with $\sigma\tau$, at any rate for light of sufficiently long period. In that case we can simply put, in accordance with eqn. (11),

(13) $$\beta = \sqrt{\mu\sigma\tau} .$$

The value of α is the same ; hence the *velocity of propagation* of the wave within the metal, which by eqn. (12) must be equal to c/α, is given by

(14) $$v = \frac{c}{\alpha} = \frac{c}{\sqrt{\mu\sigma\tau}} .$$

Eqns. (13) and (14) are known as *Drude's equations,* after the investigator who first established them (1889).

According to § 42 eqn. (8), the product $c\tau$ is equal to the wave-length λ (corresponding to the relevant period in free space), and hence the first exponential expression on the right-hand side of eqn. (12) may also be written in the form

$$e^{-\frac{2\pi x\beta}{\lambda}} .$$

---

[2] This symbolical solution also covers the general case of an elliptically polarized wave. In accordance with § 39 we have then only to separate the real and imaginary terms in the solution, to multiply both by different constants (omitting the factor i), and finally to add the two products together.

The amplitude of the wave is therefore reduced to $e^{-2\pi\beta}$ of its amount, after penetrating to the depth of one wave-length. On that account the quantity β is termed the *absorption coefficent of the metal*.

We see from eqn. (14) that the velocity of propagation *depends* on the period τ, and thus *on the colour*. Moreover, as eqn. (13) shows, the absorption also depends on the colour. Light of greater wave-length is more strongly absorbed than light of smaller wave-length.

In an insulator the quantity κ in the expression (7) is equal to the quotient of the refractive index and the velocity of light. In the case of metals the quantity κ is replaced by the complex quantity κ' as determined by eqn. (10), and it is not unnatural to attribute to a metal a *complex refractive index* (n') whose value is

(15) $$n' = \alpha - i\beta .$$

In accordance with the law of refraction we then have

$$\sin\psi = \frac{\sin\varphi}{\alpha - i\beta},$$

so that $\sin\psi$ is likewise complex ; and hence it follows from Fresnel's equations that the quantities A_p' and A_s' will also be complex. In virtue of § 83 eqn. (6), we may express this fact by putting

(16) $$\begin{cases} A_p' = B_p \cdot e^{i\delta_p}, \\ A_s' = B_s \cdot e^{i\delta_s}, \end{cases}$$

where, as a comparison with § 83 eqn. (11) shows us, B_p and B_s are the (real) amplitudes with which the two mutually perpendicular components of the reflected electric wave vibrate. (In the case of total reflection, such as we were considering in § 83, B_p is simply equal to A_p and B_s to A_s.) Denoting as in § 83 the difference $(\delta_p - \delta_s)$ by δ, we find from eqn. (16)

(17) $$\frac{A_p'}{A_s'} = \frac{B_p}{B_s} \cdot e^{i\delta}.$$

On the other hand, we have by § 82 eqn. (4), which it is quite permissible to apply here,

(18) $$\frac{A_p'}{A_s'} = -\frac{A_p}{A_s} \cdot \frac{\cos(\varphi + \psi)}{\cos(\varphi - \psi)}.$$

Since this expression is in general complex, the right-hand side of eqn. (17) must also in general be complex, and, inasmuch as B_p and B_s are real quantities, δ must in general

differ from zero.   The *light reflected by a metallic surface* is hence in general *elliptically polarized*, even when the incident light is plane polarized.[3]

Let us consider the special case of *normal incidence*, in which, by § 82 eqn. (8),

$$(19) \qquad A_p{}' = A_p \frac{n' - 1}{n' + 1},$$

or, replacing n' and $A_p{}'$ by their values as given in eqns. (15) and (16),

$$(20) \qquad B_p \cdot e^{i\delta_p} = A_p \frac{\alpha - i\beta - 1}{\alpha - i\beta + 1}.$$

In order to calculate the quantity $B_p{}^2$, let us write down the equation obtained by substituting the factor $(-i)$ for the factor i throughout eqn. (20).   In accordance with a well-known algebraical theorem, this equation must also be satisfied just as is eqn. (20) itself.   (For a complex equation can only hold good if the real terms on either side are equal to each other, and likewise the imaginary terms.   Thus, if we have an equation

$$A + iB = C + iD,$$

the equation

$$A - iB = C - iD$$

must also be satisfied, *etc.*)   Hence we find

$$(21) \qquad B_p \cdot e^{-i\delta_p} = A_p \frac{\alpha + i\beta - 1}{\alpha + i\beta + 1},$$

and, on multiplying eqns. (20) and (21) together, we obtain

$$(22) \qquad B_p{}^2 = A_p{}^2 \frac{\alpha^2 + \beta^2 - 2\alpha + 1}{\alpha^2 + \beta^2 + 2\alpha + 1}.$$

[3] The special angle for which δ is equal to $\pi/2$ is called the *principal angle of incidence.*   By getting rid of the difference in phase between the two components of the reflected ray (by means of a compensator) we obtain plane polarized light.   The plane of polarization of the latter, however, will, in general, make a different angle with the incident plane from that made by the plane of polarization of the incident ray.   When the incident ray strikes the reflecting surface at the principal angle of incidence, and is plane polarized at an azimuth of 45° with respect to the incident plane, the angle included between the plane of polarization of the reflected ray (the difference in phase having been removed) and the incident plane is termed the *principal azimuth.*   The principal angle of incidence and the principal azimuth are connected with the optical constants a and β by two equations, into which we shall not enter here.   It is therefore possible to determine the optical constants of a metal from measurements of the principal angle of incidence and the principal azimuth, which can be very accurately carried out.

Again, by § 82 eqn. (9)—in complete analogy with eqn. (19)—

(23)
$$A_s' = - A_s \frac{n' - 1}{n' + 1},$$

whence

(24)
$$B_s{}^2 = A_s{}^2 \frac{\alpha^2 + \beta^2 - 2\alpha + 1}{\alpha^2 + \beta^2 + 2\alpha + 1}.$$

Now the reflective power is given by

$$\varrho = \frac{B_p{}^2 + B_s{}^2}{A_p{}^2 + A_s{}^2},$$

or, by eqns. (22) and (24),

(25)
$$\varrho = \frac{\alpha^2 + \beta^2 - 2\alpha + 1}{\alpha^2 + \beta^2 + 2\alpha + 1}.$$

Since β can be put equal to α [according to eqns. (13) and (14)], we may also write

$$\varrho = 1 - \frac{4\alpha}{2\alpha^2 + 2\alpha + 1}.$$

But, as we have already explained with reference to eqn. (13), α is large as compared with unity, so that $2\alpha^2$ is large relatively to the expression $(2\alpha + 1)$. Hence we may put

(26)
$$\varrho = 1 - \frac{2}{\alpha},$$

or, by eqn. (14),

(27)
$$\varrho = 1 - \frac{2}{\sqrt{\mu\sigma\tau}}.$$

Measurements by *Rubens* and *Hagen* (1903) have shown that this equation does actually hold exactly in the case of rays of very *long wave-length*.

It follows from eqn. (26) that, as α is large in value, the *reflective power of metals differs only slightly from unity*. This property of metals is known as *metallic lustre*.[4] Moreover, as eqn. (27) shows, ϱ depends on the value of τ, so that the reflective power differs for *different colours*. Since α and β are equal in value, we can also write eqn. (26) in the form

(28)
$$\varrho = 1 - \frac{2}{\beta};$$

and we then see that colours which are strongly absorbed must also be particularly strongly reflected. The *colours* shown by metallic films for transmitted and for reflected light are therefore *complementary*.

[4] As metallic lustre depends on the ratio of the intensities of the reflected and the incident light, it is also to be observed in the case of total reflection ; for that reason air bubbles under water have the appearance of (metallic) mercury

§ 85. The Propagation of Light in Crystals.

From observations on Iceland spar (or calcite) made by *Bartholinus* in 1669, and more accurately by *Huygens* soon afterwards, it was quite evident that the simple laws holding for the propagation of light in glass, for instance, were *not sufficient* for a theory of the optical behaviour of *crystals*. *Huygens* therefore sought an explanation of the phenomena to be observed with calcite, on the basis of the assumption that the optical properties of the crystal *differ in different directions*, or, in other words, that Iceland spar is *optically æolotropic*. In Huygens' opinion, this would manifest itself in the existence of different rates of propagation of light in different directions through the crystal; and, as a matter of fact, he was able to give a simple explanation of double refraction on the basis of this hypothesis.

Now in accordance with the electromagnetic theory of light, the velocity of light in a transparent medium appears to be determined by the value of the *dielectric constant*. Consequently, if we are to introduce Huygens' assumption into the electromagnetic theory of light, we must *extend* the conception of the dielectric constant in the case of crystals.

Whereas, in an isotropic body, the vector **g**, which denotes the density of the displacement current, is always parallel to the vector $\partial \mathbf{E}/\partial t$, which represents the time-rate of change of the electric strength of field,[1] in the case of a crystal we shall have to regard the vector **g** quite generally as a symmetrical *linear vector function* of the vector $\partial \mathbf{E}/\partial t$. The scalar dielectric constant is thus to be replaced by a *tensor dielectric constant*, whose principal values (ϵ_1, ϵ_2, ϵ_3) are called the *principal dielectric constants* of the crystal; the principal axes of the tensor are known as the *electric axes of symmetry* of the crystal. Experimental measurements do actually prove that, in a crystal, the electrically determined dielectric constants depend on the direction.

Using a coordinate system whose axes are electrical axes of symmetry, we have the following relations [in accordance with § 27 eqn. (6)]:—

$$(1) \quad \begin{cases} 4\pi g_x = \epsilon_1 \dfrac{\partial E_x}{\partial t}, \\[2mm] 4\pi g_y = \epsilon_2 \dfrac{\partial E_y}{\partial t}, \\[2mm] 4\pi g_z = \epsilon_3 \dfrac{\partial E_z}{\partial t}. \end{cases}$$

[1] *Vide* § 73 eqn. (5).

In accordance with the generally valid and fundamental relation of Maxwell's theory, the density of the displacement current in an insulator, multiplied by $4\pi/c$, is vectorially equal to the rotation of the magnetic strength of field, and hence the following equations must be satisfied in the case of a crystal :—

$$(2) \qquad \begin{cases} \dfrac{\epsilon_1}{c}\dfrac{\partial E_x}{\partial t} = \mathrm{rot}_x\,\mathbf{H}, \\[2mm] \dfrac{\epsilon_2}{c}\dfrac{\partial E_y}{\partial t} = \mathrm{rot}_y\,\mathbf{H}, \\[2mm] \dfrac{\epsilon_3}{c}\dfrac{\partial E_z}{\partial t} = \mathrm{rot}_z\,\mathbf{H}. \end{cases}$$

On the other hand, Maxwell's second principal equation, which states that

$$(3) \qquad \frac{\mu}{c}\frac{\partial \mathbf{H}}{\partial t} = -\,\mathrm{rot}\,\mathbf{E},$$

can be applied in its unaltered form to crystals, for it has been found unnecessary to assume a dependency of the magnetic permeability in crystals upon direction.

Differentiating the first of eqns. (2) partially with respect to time, we find

$$\frac{\epsilon_1}{c}\frac{\partial^2 E_x}{\partial t^2} = \mathrm{rot}_x\!\left(\frac{\partial \mathbf{H}}{\partial t}\right),$$

or, by eqn. (3),

$$(4) \qquad \frac{\epsilon_1 \mu}{c^2}\frac{\partial^2 E_x}{\partial t^2} = -\,\mathrm{rot}_x\,(\mathrm{rot}\,\mathbf{E}).$$

Making use of the vectorial analytical formula for the dual rotation of a vector [§ 32 eqn. (25)], we thus obtain the three equations[2]

$$(5) \qquad \begin{cases} \dfrac{\epsilon_1\,\mu}{c^2}\dfrac{\partial^2 E_x}{\partial t^2} = \nabla^2 E_x - \dfrac{\partial}{\partial x}\,(\mathrm{div}\,\mathbf{E}), \\[2mm] \dfrac{\epsilon_2\,\mu}{c^2}\dfrac{\partial^2 E_y}{\partial t^2} = \nabla^2 E_y - \dfrac{\partial}{\partial y}\,(\mathrm{div}\,\mathbf{E}), \\[2mm] \dfrac{\epsilon_3\,\mu}{c^2}\dfrac{\partial^2 E_z}{\partial t^2} = \nabla^2 E_z - \dfrac{\partial}{\partial z}\,(\mathrm{div}\,\mathbf{E}). \end{cases}$$

[2] Even when no charges are present we have no right to put div E equal to zero, for this is only permissible for isotropic media in accordance with the relation

$$\mathrm{div}\,(\epsilon \mathbf{E}) = 4\pi\varrho.$$

It is only when ϵ can be placed outside the brackets that E vanishes when $\varrho = 0$. Otherwise we must bear in mind § 32 eqn. (26).

Let us now choose any arbitrary point within the crystal as the origin of a coordinate system whose axes are parallel to the electrical axes of symmetry, and let us draw through it any straight line making angles α, β, γ with the three coordinate axes. We wish to investigate whether, and under what conditions with eqns. (5) as our basis, a *plane polarized plane wave of light* can be propagated along this straight line as *wave-normal*. Granted that such a propagation is possible, the following three equations must be satisfied :—

$$(6) \quad \begin{cases} E_x = A_1 \sin\left[\dfrac{2\pi}{\tau}\left(t - \dfrac{x\cos\alpha + y\cos\beta + z\cos\gamma}{v}\right)\right], \\[2ex] E_y = A_2 \sin\left[\dfrac{2\pi}{\tau}\left(t - \dfrac{x\cos\alpha + y\cos\beta + z\cos\gamma}{v}\right)\right], \\[2ex] E_z = A_3 \sin\left[\dfrac{2\pi}{\tau}\left(t - \dfrac{x\cos\alpha + y\cos\beta + z\cos\gamma}{v}\right)\right]. \end{cases}$$

Differentiating these equations partially, we find

$$(7) \quad \begin{cases} \dfrac{\partial E_x}{\partial x} = -A_1 \cos\left[\qquad\right]\dfrac{2\pi}{\tau}\dfrac{\cos\alpha}{v}, \\[2ex] \dfrac{\partial E_x}{\partial y} = -A_1 \cos\left[\qquad\right]\dfrac{2\pi}{\tau}\dfrac{\cos\beta}{v}, \\[2ex] \dfrac{\partial E_y}{\partial y} = -A_2 \cos\left[\qquad\right]\dfrac{2\pi}{\tau}\dfrac{\cos\beta}{v}, \ etc., \end{cases}$$

where the square brackets (left unfilled) are understood to contain the corresponding expression in eqns. (6).

A second partial differentiation results in the following equations :—

$$(8) \quad \begin{cases} \dfrac{\partial^2 E_x}{\partial x^2} = -E_x \cdot \dfrac{4\pi^2}{\tau^2}\dfrac{\cos^2\alpha}{v^2}, \\[2ex] \dfrac{\partial^2 E_x}{\partial y^2} = -E_x \cdot \dfrac{4\pi^2}{\tau^2}\dfrac{\cos^2\beta}{v^2}, \\[2ex] \dfrac{\partial^2 E_y}{\partial x\,\partial y} = -E_y \cdot \dfrac{4\pi^2}{\tau^2}\dfrac{\cos\alpha\cos\beta}{v^2}, \ etc. \end{cases}$$

Since

$$(9) \qquad \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1,$$

it follows from eqn. (8) that

$$(10) \quad \nabla^2 E_x = \dfrac{\partial^2 E_x}{\partial x^2} + \dfrac{\partial^2 E_x}{\partial y^2} + \dfrac{\partial^2 E_x}{\partial z^2} = -E_x \cdot \dfrac{4\pi^2}{\tau^2\,v^2}.$$

Again, by eqn. (8),

(11)
$$\frac{\partial}{\partial x}(\operatorname{div} \mathbf{E}) = \frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_y}{\partial x\,\partial y} + \frac{\partial^2 E_z}{\partial x\,\partial z}$$
$$= -\frac{4\pi^2 \cos\alpha}{\tau^2\,v^2}\left(E_x\cos\alpha + E_y\cos\beta + E_z\cos\gamma\right).$$

Finally,

(12)
$$\frac{\partial^2 E_x}{\partial t^2} = -E_x \cdot \frac{4\pi^2}{\tau^2}.$$

Introducing the values given by eqns. (10), (11), and (12) into eqn. (5), we see that eqns. (6) can only represent a solution of the differential eqn. (5) provided that

(13)
$$-\frac{\epsilon_1\,\mu}{c^2}\cdot\frac{4\pi^2}{\tau^2}\,E_x = -\frac{4\pi^2}{\tau^2\,v^2}\cdot E_x$$
$$+\frac{4\pi^2\cos\alpha}{\tau^2\,v^2}\left(E_x\cos\alpha + E_y\cos\beta + E_z\cos\gamma\right),$$

and that the two analogous relations for ϵ_2 and ϵ_3 also hold good.

Let us put for short

(14)
$$\frac{c^2}{\epsilon_1\,\mu} = v_1{}^2, \qquad \frac{c^2}{\epsilon_2\,\mu} = v_2{}^2, \qquad \frac{c^2}{\epsilon_3\,\mu} = v_3{}^2,$$

and [3]

(15)
$$E_x\cos\alpha + E_y\cos\beta + E_z\cos\gamma = S.$$

Then we may write eqn. (13) in the form

$$\frac{E_x}{v_1{}^2} = \frac{E_x}{v^2} - \frac{S\cos\alpha}{v^2},$$

or,

(16)
$$E_x = \frac{S v_1{}^2 \cos\alpha}{v_1{}^2 - v^2}.$$

Multiplying this equation by $\cos\alpha$ and the two analogous equations by $\cos\beta$ and $\cos\gamma$, and then adding all three together, we find, in accordance with eqn. (15) and after dividing throughout by S,

(17)
$$1 = \frac{v_1{}^2\cos^2\alpha}{v_1{}^2 - v^2} + \frac{v_2{}^2\cos^2\beta}{v_2{}^2 - v^2} + \frac{v_3{}^2\cos^2\gamma}{v_3{}^2 - v^2}.$$

We will now use the identity

$$\frac{v_1{}^2\cos^2\alpha}{v_1{}^2 - v^2} = \cos^2\alpha + \frac{v^2\cos^2\alpha}{v_1{}^2 - v^2},$$

[3] v_1 is therefore equal to the velocity of light in an isotropic medium whose dielectric constant is ϵ_1, etc.

together with the two analogous relations, in order to transform eqn. (17). Bearing in mind eqn. (9), and dividing throughout by v^2, we may then write eqn. (17) as

$$(18) \qquad \frac{\cos^2 \alpha}{v_1{}^2 - v^2} + \frac{\cos^2 \beta}{v_2{}^2 - v^2} + \frac{\cos^2 \gamma}{v_3{}^2 - v^2} = 0 \,.$$

This formula is an expression of *Fresnel's law of the propagation of light in crystals.*

We can also write eqn. (18) in the form

$$(19) \quad \left\{ \begin{array}{c} \cos^2 \alpha \, (v_2{}^2 - v^2) \, (v_3{}^2 - v^2) + \cos^2 \beta \, (v_3{}^2 - v^2)(v_1{}^2 - v^2) \\ + \cos^2 \gamma \, (v_1{}^2 - v^2) \, (v_2{}^2 - v^2) = 0. \end{array} \right.$$

We see that this equation is of the *second degree* as regards the unknown v^2. Hence there are *two different solutions* [4] for every direction as defined by the values of α, β, γ. If, for instance, the wave-normal coincides with the direction of x, so that

$$\cos \alpha = 1 \,, \qquad \cos \beta = 0 \,, \qquad \cos \gamma = 0 \,,$$

the two solutions are

$$v' = v_2 \,; \qquad v'' = v_3.$$

Thus every straight line parallel to the x-axis can form the wave-normal of *two different waves* propagated with velocities v_2 and v_3. Similarly, there correspond to the y-axis as wave-normal, velocities of propagation v_3 and v_1; and to the z-axis, velocities v_1 and v_2. The three constants v_1, v_2, v_3 are consequently termed the *principal velocities of light* in the crystal. Quite generally, any arbitrary direction is associated with two velocities of propagation whose connection with the principal velocities is given by eqn. (18).

Now in the case of isotropic media, as we have already seen in § 73, the product $\epsilon \mathbf{E}$ represents a vector which is independent of the nature of the medium and equal to the electric strength of field in free space. In the case of crystals, the same will hold good for a vector (**D**) which is a linear vector function of the electric field-strength and the tensor dielectric constant, and the values of its components relatively to a system of principal axes are therefore

$$(20) \qquad D_x = \epsilon_1 \, E_x \,, \qquad D_y = \epsilon_2 \, E_y \,, \qquad D_z = \epsilon_3 \, E_z \,.$$

[4] It is true that there are actually four solutions for v, but these are equal by pairs, apart from the sign which has no meaning in a physical sense.

It follows from eqns. (14) and (16) that

$$(21) \quad \begin{cases} D_x = \dfrac{Sc^2}{\mu} \dfrac{\cos \alpha}{v_1{}^2 - v^2}, \\[2mm] D_y = \dfrac{Sc^2}{\mu} \dfrac{\cos \beta}{v_2{}^2 - v^2}, \\[2mm] D_z = \dfrac{Sc^2}{\mu} \dfrac{\cos \gamma}{v_3{}^2 - v^2}. \end{cases}$$

If we now denote by $\mathfrak{f}$ the unit-vector lying in the direction of the wave-normal, its components will be $\cos \alpha$, $\cos \beta$, $\cos \gamma$. Hence the scalar product of this vector and the vector $\mathbf{D}$ will be

$$(22) \qquad \mathbf{D}\mathfrak{f} = D_x \cos \alpha + D_y \cos \beta + D_z \cos \gamma.$$

Multiplying the three eqns. (21) by $\cos \alpha$, $\cos \beta$, and $\cos \gamma$, and then adding all three together, we find, in consequence of eqn. (18),

$$(23) \qquad\qquad \mathbf{D}\mathfrak{f} = 0, \text{ or } \mathbf{D} \perp \mathfrak{f}.$$

Different values of the vector $\mathbf{D}$ will naturally correspond to the two different waves which are propagated along any given straight line as wave-normal; let us denote these values by $\mathbf{D}'$ and $\mathbf{D}''$, and the velocities of the two waves by v' and v''. We then have by eqn. (21)

$$(24) \quad \begin{cases} D_x' D_x'' + D_y' D_y'' + D_z' D_z'' \\[2mm] \propto \dfrac{\cos^2 \alpha}{(v_1{}^2 - v'^2)(v_1{}^2 - v''^2)} + \dfrac{\cos^2 \beta}{(v_2{}^2 - v'^2)(v_2{}^2 - v''^2)} \\[2mm] \qquad + \dfrac{\cos^2 \gamma}{(v_3{}^2 - v'^2)(v_3{}^2 - v''^2)}. \end{cases}$$

Using the identity

$$(25) \qquad \dfrac{1}{v_1{}^2 - v'^2} - \dfrac{1}{v_1{}^2 - v''^2} = \dfrac{v'^2 - v''^2}{(v_1{}^2 - v'^2)(v_1{}^2 - v''^2)},$$

we can transform this relation, and thereby obtain

$$(26) \quad \begin{cases} \mathbf{D}'\mathbf{D}'' \propto \dfrac{1}{v'^2 - v''^2} \left\{ \left[\dfrac{\cos^2 \alpha}{v_1{}^2 - v'^2} + \dfrac{\cos^2 \beta}{v_2{}^2 - v'^2} + \dfrac{\cos^2 \gamma}{v_3{}^2 - v'^2} \right] \right. \\[2mm] \qquad \left. - \left[\dfrac{\cos^2 \alpha}{v_1{}^2 - v''^2} + \dfrac{\cos^2 \beta}{v_2{}^2 - v''^2} + \dfrac{\cos^2 \gamma}{v_3{}^2 - v''^2} \right] \right\}. \end{cases}$$

The expressions in both square brackets must vanish, however, seeing that eqn. (18) must be satisfied both for $v = v'$ and for $v = v''$, and since v' and v'' are the actual roots o this equation. Hence we find

$$(27) \qquad\qquad \mathbf{D}'\mathbf{D}'' = 0, \text{ or } \mathbf{D}' \perp \mathbf{D}''.$$

If, therefore, we understand the electric vibrations to be the vibrations of the vector **D**, we may say that the two waves belonging to the same wave-normal in a crystal are *polarized at right angles to each other*.[5]

When *two of the three principal velocities of light are equal*, the crystal is termed *uniaxial*.[6] If, for example, $v_3 = v_2$, both waves are transmitted in the direction of x with the same velocity ; in that direction there is only a single wave, just as in the case of an isotropic body. The x-axis, or any straight line parallel to it, is then known as the *principal optical axis of the crystal* ; a plane determined by the wave-normal and by a principal optical axis which intersects the normal is termed a *principal plane* or *principal section* with respect to that normal.

When $v_3 = v_2$ we find, since

(28) $\cos^2 \beta + \cos^2 \gamma = 1 - \cos^2 \alpha = \sin^2 \alpha$,

that eqn. (19) takes the simple form

(29) $\cos^2 \alpha (v_2^2 - v^2)^2 + \sin^2 \alpha (v_1^2 - v^2)(v_2^2 - v^2) = 0$.

This equation has two solutions, *viz.*, in the first place,

(30) $v'^2 = v_2^2$;

and secondly [on dividing eqn. (29) by $(v_2^2 - v^2)$],

(31) $\cos^2 \alpha (v_2^2 - v''^2) + \sin^2 \alpha (v_1^2 - v''^2) = 0$,

whence

(32) $v''^2 = v_2^2 \cos^2 \alpha + v_1^2 \sin^2 \alpha$.

Eqn. (30) shows us that any arbitrarily directed wave-normal in a uniaxial crystal is associated with a wave which is transmitted with one and the same velocity, independently of the direction, just as though the crystal were isotropic. This wave is called the *ordinary wave*, and its velocity is equal to the single velocity along the principal axis. The velocity of the second wave, which, in contradistinction, is termed the *extraordinary wave*, depends on the direction of the wave-normal, but is completely determined by the *inclination* of

[5] We cannot enter into the proof here, but it may be shown that, in any case, *after leaving the crystal* the two rays are polarized at right angles to each other, *i.e.*, in free space or in an isotropic medium where **E** and **D** are again similarly directed. This is quite independent of whether the vibrations of the vector **D** or of the vector **E** (which is, in general, differently directed) are taken as the electric vibrations in the crystal.

[6] Crystals of the *tetragonal* and *hexagonal* systems are uniaxial. When $v_2 > v_1$, as in quartz, the crystal is said to be *positively* uniaxial; when $v_2 < v_1$, as in calcite, it is called *negatively* uniaxial.—Crystals of the regular, cubic system cannot be distinguished at all, in an optical respect, from isotropic bodies.

that normal relatively to the principal optical axis. This angle of inclination is, in fact, equal to α, because we have made the principal optical axis coincide with the x-axis. When $\alpha = 90°$, the velocity of the extraordinary wave is equal to v_1.[7]

The directions in which the electric vibrations take place in the two waves can be derived from eqns. (21), for it follows from these that in the case of a uniaxial crystal, on account of the equality of v_2 and v_3, we have the proportion

$$(33) \qquad D_x : D_y : D_z = \cos \alpha \, \frac{v_2{}^2 - v^2}{v_1{}^2 - v^2} : \cos \beta : \cos \gamma \,.$$

In the ordinary ray, where v is always equal to v_2, we thus have

$$(34) \qquad\qquad\qquad D_x{}' = 0 \,.$$

Hence $\mathbf{D}'$ must be perpendicular, not only to the principal optical axis (for this has the same direction as the x-axis), but also [by eqn. (23)] to the wave-normal. The *electric vibrations in the ordinary wave* therefore take place *at right angles to the principal plane*. Since, however, according to eqn. (27) the ordinary and the extraordinary waves are polarized at right angles to each other, the electric vibrations in the *extraordinary wave* must take place *in the principal plane*.

Now no wave-normal can ever be associated with more than two waves. Hence *only* those waves whose electric vibrations occur either in a principal plane or perpendicular to it can be transmitted through a uniaxial crystal. In the second case, the propagation of the waves through a uniaxial crystal is the same as through an isotropic medium.[8]

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[7] It may be shown (but we shall not enter into the proof here) that the *elementary waves* of the ordinary ray are *spherical surfaces*, while those of the extraordinary ray are *ellipsoids of rotation*, and that the two waves touch each other in the direction of the principal optical axis. On this fact are based the well-known constructions used in experimental physics, and due to Huygens and Fresnel, which require an extension for biaxial crystals.

[8] It may be deduced from the electromagnetic theory of light that in an *absorbing crystal*, in which we have to regard the *conductivity* as differing from zero, and as itself being a symmetrical *tensor*, the absorption of a wave depends on its *state of polarization*. This explains the fact that uniaxial *tourmaline* absorbs the ordinary ray almost completely while allowing the extraordinary ray to pass through. A tourmaline plate can thus be used to produce polarized light.

APPENDIX

Chapter I

The Motion of a Free Material Particle

§ 1. In every motion the instantaneous value of the velocity is determined by the differential coefficient ds/dt, where ds denotes the element of path traversed in the element of time dt. The direction of the velocity at any moment is given by the direction of the path at that moment. As we learn from the Principle of Inertia, every body maintains the value of its velocity and its direction of motion unaltered, provided that its motion be not affected by an external cause known as force. Since, as experience shows, the action of every force can be annulled by the pull of a definite weight exerted in a definite direction, we must ascribe a definite magnitude and direction to every force. Hence forces may be symbolically represented by directed lines.

§ 2. Every vector quantity is determined by a magnitude, a direction, and a sense; the fact that two vectors agree in these properties is expressed by equating the quantities to each other. The square of the magnitude of a vector is equal to the sum of the squares of its components. Every vector may be regarded as the product of its magnitude and a unit-vector lying in its direction. The unit-vectors corresponding to the positive coordinate axes are called the fundamental vectors (i, j, k) of the coordinate system.

§ 3. Vectorial addition is governed by the same rules as

arithmetical. The internal or scalar product **AB** is a scalar whose magnitude is $AB \cos(\mathbf{A}, \mathbf{B})$. We have

$$\mathbf{AB} = A_x B_x + A_y B_y + A_z B_z.$$

The external or vector product [**AB**] is a vector which is perpendicular to the plane of the parallelogram formed by the vectors **A** and **B**, which contains as many units of length as does the parallelogram units of area, and whose sense is such that, when seen from its extremity, the rotation necessary to carry the vector **A** into the direction of the vector **B** appears anti-clockwise. [**BA**] is equal to [**AB**] in magnitude and direction, but opposite in sense. The scalar product of two mutually perpendicular vectors vanishes, as does the vector product of two similarly directed vectors. We have

$$\mathbf{A}[\mathbf{BC}] = \mathbf{B}[\mathbf{CA}] = \mathbf{C}[\mathbf{AB}];$$

and these products vanish when the vectors **A**, **B**, **C** are co-planar. Again

$$[\mathbf{A}[\mathbf{BC}]] = \mathbf{B}(\mathbf{CA}) - \mathbf{C}(\mathbf{AB}).$$

The time-derivative of a vector **A** can always be resolved into two perpendicular components, one of which lies in the direction of the vector and is of magnitude dA/dt, whilst the other is at right angles to it and of magnitude $A\, d\varphi/dt$, where $d\varphi$ is the angle through which the vector **A** turns in the interval dt. Both components lie in the surface traversed by the vector during that interval, this surface being regarded as plane.

§ 4. As we learn from Newton's Second Law of Motion, the vector of force is always proportional to the time-rate of change of the velocity vector (called the acceleration). The proportionality factor is defined as the mass of the body. An acceleration can be resolved into a tangential and a normal component given by the formulæ

$$b_t = \frac{dv}{dt}, \qquad b_n = \frac{v^2}{\varrho},$$

where ϱ denotes the radius of curvature of the path. The product of the mass and the velocity vector is known as the momentum, and its time-derivative is equal to the force. When several forces act simultaneously on a particle, the product of the mass and the actual acceleration is equal to the resultant force, which is obtained by vectorial addition of the individual forces.

§ 5. From the equations of motion we can derive the laws of projectile motion. The trajectory proves to be a parabola. The maximum range is attained with an angle

of elevation of 45°, while, in general, two different values of the angle of elevation correspond to every range. In the case of free fall, the distance (h), the velocity (v) of fall, and the acceleration (g) due to gravity are connected by the relations

$$v = \sqrt{2gh}, \qquad h = \frac{1}{2}gt^2.$$

In a resisting medium the velocity of fall approaches, in time, a constant limit g/k, where k denotes the quotient of the coefficient of frictional resistance and the mass of the falling body.

§ 6. For small angular displacements, the value of the latter in a pendulum motion is a sine function of the time. The period of oscillation (τ) is given by the relation

$$\tau = 2\pi \sqrt{\frac{l}{g}},$$

where l is the length of the pendulum.

§ 7. The vector products obtained on multiplying vectorially the radius-vector drawn from any point by the force, the velocity, or the momentum are known respectively as the moment of force, the moment of velocity, and the moment of momentum (or the angular momentum). The moment of force is the time-rate of change of the angular momentum. It follows that, when a body is acted on by a force which is always directed towards the same point, the motion of the body continues in one and the same plane, and its moment of velocity about that point does not vary.

§ 8. A purely mathematical derivation from Kepler's laws of planetary motion shows us that the acceleration of every planet is always directed towards the sun, and that it is completely independent of all individual properties of the planet. It is, in fact, only a function of the distance of the planet from the sun, and is inversely proportional to the square of this distance. From this follows Newton's Law of Gravitation, which states that the force of attraction between two bodies of mass m_1 and m_2, placed at a distance r apart, is equal to $\kappa\, m_1 m_2/r^2$, where κ denotes the constant of gravitation ($6\cdot68 \times 10^{-8}$ gram^{-1} cm.^{3} sec.$^{-2}$).

§ 9. In changing from one system of coordinates to another, the components of a vector are transformed in accordance with the scheme

$$A_{x'} = A_x \cos(x', x) + A_y \cos(x', y) + A_z \cos(x', z).$$

§ 10. In a scalar field every point can be associated with a vector, known as the gradient of the scalar, which is per-

pendicular to the field-level, whose sense is that of the direction in which the scalar increases, and whose magnitude is equal to the linear rate of increase of the scalar in that direction. The components of the gradient are equal to the partial derivatives of the scalar with respect to the coordinates. The value of the line-integral of the gradient along a curve depends simply and solely on the difference in level of the scalar between the initial and final points of the curve. In particular,

$$\operatorname{grad}\left(\frac{1}{r}\right) = -\frac{1}{r^2}\frac{\mathbf{r}}{r},$$

where $\mathbf{r}$ is the radius-vector drawn from a fixed point to any arbitrary point in the field.

§ 11. When a vector can be represented as the gradient of a scalar, this scalar, taken with the opposite sign, is called the potential of the vector. A force will thus always be associated with a potential when it is directed towards a fixed point, and when its magnitude depends only on its distance from the latter. Where the force can be derived from a potential, the sum of this potential and the kinetic energy of the moving particle must represent a constant independent of place and time, which is known as the mechanical energy of the moving particle.

§ 12. If a coordinate system ($\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$) changes in position relatively to an assumed standard system, the fundamental vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ will themselves appear as functions of the time with respect to this standard system. It may be shown that the three vectors $d\mathbf{i}/dt$, $d\mathbf{j}/dt$, $d\mathbf{k}/dt$ are co-planar, and moreover, that they can be represented as vector products $[\mathbf{wi}]$, $[\mathbf{wj}]$, $[\mathbf{wk}]$, where $\mathbf{w}$ denotes one and the same vector, known as the angular velocity. The direction and sense of this vector are determined by the condition that any point rigidly connected with the coordinate system revolves in a plane perpendicular to the direction of $\mathbf{w}$, in such a way that, when viewed from the extremity of the vector, the revolution appears anti-clockwise. Moreover, the magnitude of the vector is equal to the velocity of rotation (measured in radians) of a perpendicular dropped from any point rigidly connected with the coordinate system upon the direction of $\mathbf{w}$. If $d\mathbf{A}/dt$ be the time-derivative of a vector with respect to the standard system, and $d^*\mathbf{A}/dt$ that with respect to the moving system itself, we have

$$\frac{d\mathbf{A}}{dt} = \frac{d^*\mathbf{A}}{dt} + [\mathbf{wA}].$$

§ 13. We find that an absolute acceleration is equal to the vectorial sum of the relative acceleration, the acceleration of transport, and the Coriolis acceleration, the latter, in its turn, being equal to twice the vector product of the angular and relative velocities. In referring the motion of a particle to a system which is itself in motion, we therefore have to assume the existence of two imaginary accessory forces in addition to the forces actually at work, these imaginary forces being a force of transport and a Coriolis force. The equations of motion will have the same form for two systems in a state of uniform translatory motion with respect to each other (the mechanical principle of relativity).

§ 14. From the theory of relative motion we can derive the following conclusions, referring to motion on the rotating earth. In the case of free fall there will be an easterly deviation ; a horizontal motion will be subjected to a deflection towards the right in the northern hemisphere, and towards the left in the southern ; and, finally, the plane of oscillation of an oscillating pendulum will rotate daily through an angle of 360° multiplied by the sine of the geographical latitude.

Chapter II

The General Principles of Mechanics

§ 15. It follows from Newton's Third Law of Motion that, in a system of free particles, the time-derivative of the total momentum must be equal to the resultant of all the forces acting on the system. Hence, in the absence of external forces, the total momentum remains unchanged. The system always behaves as though its whole mass were concentrated at the centre of mass (or centre of gravity), and as if all the forces were acting at that point.

§ 16. Provided that the internal forces in a system are central forces, and that no external forces are acting, the total angular momentum of the system with respect to any given point is constant both in magnitude and direction. When the centre of gravity of the system is at rest, the angular momentum is also independent of the position of the reference point.

§ 17. When the internal forces are central forces and functions only of the mutual distances, and when also the external forces can be derived from a potential, the sum of the external and internal potential energies and of the kinetic energy remains constant ; this sum is known as the mechanical energy.

§ 18. A system of particles whose freedom of motion is subjected to arbitrary limitations will be in equilibrium when the relation

$$\sum \mathbf{K}_h \, \delta \mathbf{s}_h = 0$$

is satisfied ; $\mathbf{K}_h$ denotes the external force acting on the hth particle, and $\delta \mathbf{s}_h$ the displacement experienced by the particle for a virtual change in the configuration of the system.

§ 19. The problem of restricted motion can be reduced to the question of equilibrium, with the aid of d'Alembert's Principle. For, if a particle whose freedom of motion is limited is moving under the influence of a force $\mathbf{K}$ with an instantaneous actual acceleration $\mathbf{b}$, we may assume the existence of an accessory force such that the vectorial sum of it and the force $\mathbf{K}$ actually operating is equal to the product of the mass and the actual acceleration $\mathbf{b}$. In accordance with d'Alembert's Principle, the accessory forces which we have to imagine acting on the individual particles of a system in this way (or the lost forces which are defined to be equal and opposite to the accessory forces) must be in equilibrium. Hence the general formula of equilibrium (of § 18) must be applicable to them. The combination of d'Alembert's Principle with the Principle of Virtual Displacements leads to Lagrange's general formula of motion :—

$$\sum_{h=1}^{h=n} \left\{ \left(X_h - m_h \frac{d^2 x_h}{dt^2} \right) \delta x_h + \left(Y_h - m_h \frac{d^2 y_h}{dt^2} \right) \delta y_h \\ + \left(Z_h - m_h \frac{d^2 z_h}{dt^2} \right) \delta z_h \right\} = 0 \, .$$

§ 20. By means of Lagrange's method of undetermined multipliers, we can derive from the general formula of motion the special equations of motion which describe the motions of the individual particles. If the freedom of motion of a system of n particles be restricted by m equations of condition between the $3n$ coordinates of the system, the number $(3n - m)$ is called the number of degrees of freedom of the system.

§ 21. We obtain Hamilton's Principle from the general formula of motion by comparing the actual motion of a system with an imaginary motion which deviates infinitesimally from it, and which likewise satisfies the conditions prescribed for the system, the time, however, not being varied in the variation of the motion. Comparing the values of the kinetic and potential energies in the actual and the

varied motion, we find as an expression of Hamilton's Principle

$$\int_{t_0}^{t_1} \delta\,(L - V)\,dt = 0\,.$$

It is, however, an essential condition that the position of the system remains unvaried at the times chosen as integration limits.

§ 22. Lagrange's equations of motion in generalized coordinates, which may be entirely arbitrary functions of the $3n$ rectangular coordinates, can be derived from Hamilton's Principle. If, in particular, we choose the number of generalized coordinates (q_i) as being equal to that of the degrees of freedom of the system (s), so that the s generalized coordinates can be regarded as being mutually independent, the generalized equations of motion become

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = -\frac{\partial V}{\partial q_i}\ (i = 1 \text{ to } s)\,.$$

The partial derivatives of the kinetic energy with respect to the generalized velocities are known as the generalized momenta. The so-called canonical equations may be derived from the generalized equations of motion by introducing the Hamiltonian function, which represents the energy as a function of the generalized coordinates and momenta.

CHAPTER III

The Motion of Rigid Bodies

§ 23. A system of discrete particles whose mutual distances are invariable represents a rigid body. It may be shown that the rigid connections are equivalent to imaginary internal central forces, and hence the Principles both of the Centre of Mass and of the Conservation of the total Angular Momentum can be applied to a rigid body. If the total momentum of such a body be **G**, and the angular momentum with respect to an arbitrary reference point be **U**, the following equations must thus be satisfied

$$\frac{d\mathbf{G}}{dt} = \sum \mathbf{K}, \qquad \frac{d\mathbf{U}}{dt} = \sum [\mathbf{rK}]\,,$$

where **r** is the radius-vector drawn from the reference point to any particle in the rigid body.

§ 24. It may be easily shown that, at any given instant, the vector of angular velocity has one and the same value for all possible coordinate systems which are fixed to a rigid body. Hence we can always regard the motion of a rigid

body at any instant to be compounded of a translation which takes place with the actual velocity and direction of motion of a point fixed to the body, and of an instantaneous rotation about an axis passing through this point and also rigidly connected with the body. Thus, by a suitable choice of the reference point, any infinitesimal change in position of a rigid body may be regarded as an element of a screw motion. The equations of motion of a rigid body assume an especially simple form when we identify the arbitrary reference point with the centre of gravity.

§ 25. The whole of the forces acting on a rigid body can always be compounded into a resultant single force and a resultant couple. A straight line may always be determined such that, when a point on it is chosen as reference point, the resultant moment and the resultant single force have the same direction.

§ 26. Nine quantities k_{xx}, k_{xy} . . . k_{zz} are defined to be the components of a tensor if, on a change of the coordinate system, they undergo the same transformations as do the products formed by multiplying the components of two vectors together in pairs. The sum of the tensor components of the first kind ($k_{xx} + k_{yy} + k_{zz}$) represents a scalar, whilst the differences of the components of the second kind taken by pairs ($k_{xy} - k_{yx}$, etc.) yield the components of a vector. By a suitable multiplication of the components of a tensor by the components of a vector **B**, we can derive a vector **A** according to the scheme

$$k_{xx}B_x + k_{xy}B_y + k_{xz}B_z = A_x.$$

A is then a linear vector function of **B** and k. By the addition of corresponding components of tensors we again obtain the components of a tensor, as we also do on increasing the components of the first kind by one and the same scalar, without altering the components of the second kind.

§ 27. Every tensor can be represented by a surface of the second degree which, in particular, is an ellipsoid when the tensor components of the first kind can never become negative. The directions determined by the axes of this tensor ellipsoid are called the principal axes of the tensor ; the values assumed by the components of the first kind relative to these axes are termed the principal values of the tensor. If the tensor is symmetrical, the components of the second kind vanish with respect to a system of principal axes. If a vector **A** is a symmetrical vector function of a second vector **B** and a tensor k, the principal values of the tensor are con-

nected with the components relative to the principal axes by the relations

$$A_1 = k_1 B_1, \qquad A_2 = k_2 B_2, \qquad A_3 = k_3 B_3.$$

§ 28. The angular momentum of a rigid body proves to be a linear vector function of the angular velocity and a symmetrical tensor which is known as the moment of inertia. The moment of inertia about an arbitrary axis is given by the expression $\sum mr^2$, where r is the distance from the axis.

By Steiner's Theorem, the moment of inertia about any given axis is equal to the moment about a parallel axis which passes through the centre of gravity, increased by the product of the total mass of the body and the square of the distance of the axis in question from the centre of gravity.

§ 29. In a rigid body, the kinetic energy of rotation is equal to half the product of the moment of inertia about the axis of rotation and the square of the angular velocity. The moment of the resultant couple vanishes when the axis of rotation coincides with a principal axis of inertia. When such is not the case the resultant moment tends to deflect the axis of rotation, and the components of the second kind in the tensor of the moment of inertia are therefore called moments of deviation.

§ 30. If we refer the motion of a rigid body to a coordinate system formed by the three principal axes of inertia, with the centre of gravity as origin, we obtain Euler's equations of motion of a rigid body. It appears from these that the principal axes of inertia are also the free axes of a rigid body.

§ 31. A compound pendulum oscillates in the same manner as a simple pendulum of length I/ma, where I is the moment of inertia, m the mass of the pendulum, and a the distance between the axis of rotation and the centre of gravity. To every value of a there corresponds in general a second, different value for which the period of oscillation is the same as in the first case. By measuring this second value and the period, we can calculate for the earth the acceleration due to gravity, with the help of a simple relation.

CHAPTER IV

General Theory of Vector Fields

§ 32. The nine partial differential coefficients of the components of a vector with respect to the coordinates represent

the components of a tensor. It follows at once from the general theory of tensors that every point in a vector field may be associated with a scalar, the " divergence," and with a vector, the " rotation." We have the equations

$$\operatorname{div} \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z},$$

$$\operatorname{rot}_x \mathbf{A} = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}.$$

On multiplying the nine partial derivatives of a vector **A** with respect to the coordinates by the components of a vector **B**, in accordance with the corresponding scheme in § 26, we obtain a vector which is known as the gradient of the vector **A** relative to the vector **B**. We have

$$\left\{ \begin{array}{l} (\mathbf{B}\operatorname{grad})\,\mathbf{A} = \mathbf{i}\,(\mathbf{B}\operatorname{grad} A_x) + \mathbf{j}\,(\mathbf{B}\operatorname{grad} A_y) \\ \qquad\qquad + \mathbf{k}\,(\mathbf{B}\operatorname{grad} A_z). \end{array} \right.$$

Among the combined differential operations of vector analysis we have the formulæ

$$\operatorname{div}\operatorname{grad} S = \nabla^2 S = \frac{\partial^2 S}{\partial x^2} + \frac{\partial^2 S}{\partial y^2} + \frac{\partial^2 S}{\partial z^2}.$$

Correspondingly,

$$\nabla^2 \mathbf{A} = \mathbf{i}\,\nabla^2 A_x + \mathbf{j}\,\nabla^2 A_y + \mathbf{k}\,\nabla^2 A_z.$$

Moreover

$$\operatorname{rot}\operatorname{grad} S = 0, \qquad \operatorname{div}\operatorname{rot} \mathbf{A} = 0,$$

$$\operatorname{rot}\operatorname{rot} \mathbf{A} = \operatorname{grad}\operatorname{div} \mathbf{A} - \nabla^2 \mathbf{A}.$$

§ 33. By Gauss's theorem, the flux of a vector through a closed surface is equal to the integral of the divergence of the vector, taken over the whole volume enclosed by the surface, or

$$\int A_n\, df = \int \operatorname{div} \mathbf{A}\, d\tau.$$

Gauss's theorem leads to Green's theorem, according to which, in particular, when the gradient of a scalar vanishes at the boundaries of a scalar field, the following relation exists :—

$$\int S\,\nabla^2 S\, d\tau = - \int (\operatorname{grad} S)^2\, d\tau.$$

§ 34. A curve constructed in a vector field such that the elements of the curve have everywhere the same direction as the vector is known as a vector line. In a field where the divergence vanishes throughout, and which is termed sole-

noidal, the product of the magnitude of the vector and the cross-section of a vector tube is constant along the tube. Hence the tubes of a solenoidal vector can neither begin nor terminate within the field, but must represent closed curves in it. The field of a vector which, in its turn, is the gradient of a scalar can be represented by the field-levels of the scalar : the magnitude of the vector is everywhere proportional to the thickness of the lamellæ thus formed.

§ 35. According to Stokes's theorem, the line-integral of a vector along a closed curve is equal to the integral of the normal component of the rotation of the vector, taken over the whole surface bounded by the curve, or

$$\int \mathbf{A} \, d\mathbf{s} = \int \mathbf{n} \operatorname{rot} \mathbf{A} \, df.$$

The sense of the normal to the surface must be such that, when viewed from its extremity, the sense of formation of the line-integral appears anti-clockwise.

§ 36. For every point in a tensor field we can derive a vector known as the vector divergence (**div** t), which is given by

$$\mathbf{div}_x \, t = \frac{\partial t_{xx}}{\partial x} + \frac{\partial t_{xy}}{\partial y} + \frac{\partial t_{xz}}{\partial z}.$$

By the normal component of a tensor relative to a surface-element we mean a vector whose x-component is

$$(\mathbf{t}_n)_x = t_{xx} \cos(\mathbf{n}, x) + t_{xy} \cos(\mathbf{n}, y) + t_{xz} \cos(\mathbf{n}, z).$$

We then find that the flux of a tensor through a closed surface is equal to the volume-integral of the vector divergence of the tensor.

CHAPTER V

General Theory of Vibrations and of Waves

§ 37. The equation

$$S = A \sin\left(\frac{2\pi t}{\tau} + \epsilon\right)$$

describes a linear harmonic vibration whose amplitude is A whose period is τ, and whose phase-constant is ϵ.

§ 38. The general solution of the differential equation

$$\frac{d^2 S}{dt^2} = -a^2 S$$

is given by the expression

$$S = G \cos at + H \sin at,$$

which describes a harmonic vibration of period $2\pi/a$. In consequence of de Moivre's formula, this general integral may also be symbolically expressed by the particular integral e^{iat}.

§ 39. The general solution of the vectorial differential equation

$$\frac{d^2\mathbf{V}}{dt^2} = -a^2\mathbf{V}$$

is given by the equation

$$\mathbf{V} = \mathbf{G}\cos at + \mathbf{H}\sin at\,,$$

which describes an elliptical vibration of the vector $\mathbf{V}$. Every elliptical vibration can be resolved into two vibrations of two mutually perpendicular components of the vector which differ in phase from each other. If this difference in phase be equal to a quarter-period, and if the amplitudes of the two component vibrations are identical, the vibration will be circular.

§ 40. The differential equation

$$\frac{d^2S}{dt^2} + k\frac{dS}{dt} + a^2S = 0$$

describes a damped vibration.

§ 41. The differential equation

$$\frac{d^2S}{dt^2} + k\frac{dS}{dt} + a^2S = P\sin pt$$

represents a forced vibration. The latter is in general compounded of a diminishing characteristic vibration and of an undamped vibration whose period is that of the exciting vibration. If this period be nearly equal to that of the characteristic vibration, resonance ensues.

§ 42. The equation

$$S = A\sin\left[\frac{2\pi}{\tau}\left(t\mp\frac{x}{v}\right)+\epsilon\right],\;\text{ or } S = A\sin\left[2\pi\left(\frac{t}{\tau}\mp\frac{x}{\lambda}\right)+\epsilon\right],$$

describes a plane, harmonic, progressive wave of wavelength λ. Interference may lead to the formation of stationary waves, which are represented by the equation

$$S = 2A\cos\frac{2\pi x}{\lambda}\sin\frac{2\pi t}{\tau}.$$

The differential equation of a plane wave is

$$\frac{\partial^2 S}{\partial t^2} = \kappa^2\frac{\partial^2 S}{\partial x^2}.$$

§ 43. The differential equation

$$\frac{\partial^2 S}{\partial t^2} = \kappa^2 \nabla^2 S$$

represents a spherical wave propagated with a velocity κ. If a vectorial differential equation of the form

$$\nabla^2 \mathbf{A} = \frac{1}{\kappa^2}\frac{\partial^2 \mathbf{A}}{\partial t^2}$$

be given, it is always possible for the vector $\mathbf{A}$ to be propagated in vector waves, with the velocity κ. When, in particular, the divergence of the vector vanishes, the waves are transverse, provided they be also plane.

<center>CHAPTER VI</center>

<center>**The Motion of Deformable Bodies**</center>

§ 44. A system of particles can represent a deformable body upon three conditions. First, the particles must be so closely packed that the system can be treated as a continuously distributed mass. Secondly, it must be possible to regard the velocity as a continuous, differentiable function not only of the time, but also of the coordinates. Thirdly, the forces acting between the particles must represent central forces which, however, shall be effective only within distances of the same order of magnitude as the distance between two neighbouring particles.

§ 45. The effective tension on an arbitrary surface-element within a deformable body is found to be the normal component of a symmetrical tensor. The latter is known as the stress tensor, and its components of the first and of the second kind are distinguished as the normal and the tangential stresses.

§ 46. In a continuously distributed mass, the density of internal force is equal to the vector divergence of the stress. The acceleration of a particle in the mass is given by the relation

$$\mathbf{b} = \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\,\mathrm{grad})\,\mathbf{v}.$$

The Principle of the Conservation of Mass finds its vectorial analytical expression in the equation

$$\mathrm{div}\,(\varrho\mathbf{v}) + \frac{\partial \varrho}{\partial t} = 0.$$

The motion of a small region within a continuously distributed mass may always be regarded as the superposition of

a translation, of a rotation, and of a pure strain which appears, in general, as an extension along the coordinate axes, and as a shear along the coordinate planes.

§ 47. In an ideal fluid the tangential stresses must always vanish, and hence the normal stresses must all be equal in value, this value, taken with the opposite sign, representing the fluid pressure. In this way we may derive Euler's fundamental hydrodynamic equations from the general equation of motion of a deformable body. In the case of incompressible liquids the divergence of the velocity vanishes. On applying a vector-analytical transformation to Euler's equations, we obtain Helmholtz's equation, which expresses the conservation of vortex motion in ideal liquids upon which only such external forces act as are associated with a potential. The hydraulic pressure in a flowing liquid is always smaller than the hydrostatic. The velocity of flow within a tube must be inversely proportional to the cross-sectional area of the tube.

§ 48. A medium is defined as being elastic when the principal axes of strain coincide with the principal axes of stress, and when the relation between strain and stress can be described by Hooke's and Navier's laws. Accordingly, if we express the components of the stress in terms of the displacements, we obtain in a simple form the equations of motion of an elastic medium free from external forces.

§ 49. The equations of motion of an elastic medium establish the possibility on the one hand of longitudinal waves of dilatation, and on the other hand of transverse dilatation-free waves. In fluids only the former are possible.

Chapter VII

Theory of Potential

§ 50. At any arbitrary point in a field produced by a source whose strength is g, the potential is given by

$$\Psi = \frac{g}{r},$$

where r is the distance of the point from the source. If we define the negative gradient of the potential for a variable field-point as the strength of field, the flux of the strength of field through a closed surface is equal to $4\pi g$ or zero, according as the surface does or does not enclose the source.

§ 51. For a continuous distribution of sources we obtain Poisson's Equation, according to which

$$\nabla^2\Psi = -4\pi\varrho,$$

where ϱ denotes the density of strength of source.

§ 52. The normal component of the field-strength undergoes a sudden change equal to 4π times the surface density, when passing through a surface source. On the other hand, the tangential component of the field-strength is continuous in traversing a surface source.

§ 53. The product of the strength of a source-couple and the directed line drawn from the negative to the positive source is known as the moment of the source-couple. The potential produced by a double-layer at a field-point is equal to the product of the density of moment and the solid angle subtended at that point by the double-layer.

§ 54. By the curve potential of a curve we mean the vector

$$\mathbf{P} = \int\frac{d\mathbf{s}}{r},$$

the integration being carried out in a definite directional sense. The strength of field due to a double-layer proves to be identical with the rotation of the curve potential of the boundary of the layer, multiplied by the density of moment. The divergence of the curve potential vanishes when the curve is closed.

§ 55. It follows from Green's theorem that the density of potential energy in a field, wherein Coulomb distance forces are acting between the various sources, is equal to the square of the field-strength divided by 8π. The potential of the mechanical force exerted upon each other by two double-layers of densities of moment χ and χ' is found to be

$$V = -\chi\chi' \iint\frac{d\mathbf{s}\,d\mathbf{s}'}{r}.$$

Part II

THEORY OF THE ELECTROMAGNETIC FIELD AND OF LIGHT

Chapter VIII

Electricity and Magnetism

§ 56. Electrostatics is based on Coulomb's Law, by means of which the unit of charge, the electrostatic unit, is

defined.   Other fundamental experimental facts in electro-
statics are the dual nature of electricity and the constancy
of the algebraical sum of the whole of the quantities of
electricity in a closed system.

§ 57. Electric strength of field and electric density of
charge are connected by the fundamental relation

$$\operatorname{div} \mathbf{E} = 4\pi\varrho.$$

In an electrostatic field there exists a tension along the lines
of force, and an equally large pressure across them.   Tension
and pressure are equal in magnitude to the energy density.

§ 58. The charge on electrical conductors can only be
situated on the surface, for potential differences cannot exist
in conductors.   The potential is equal to the quotient of the
charge and the capacity.   The capacity of a spherical con-
ductor is equal to its radius.   When a conductor is placed in
an electric field, induced electricity appears on its surface ;
on this phenomenon is based the condenser.

§ 59. The divergence of the current density vanishes in
the case of closed electric currents.   The line-integral of the
electric strength of field along a conductor represents the
electromotive force.

§ 60. The fundamental law of electrostatics finds a
parallel in the following, second, experimental law :—Two
closed electric currents exert on each other a mechanical
force which is equal to the Coulomb distance force between
two shells bounded by the currents, provided that the
strength of each shell be made equal to the quotient of the
current and a universal constant, which proves on experi-
ment to be a velocity of 3×10^{10} cm./sec.   The equivalent
shell is called a magnetic shell ; and, correspondingly, we
speak of quantity of magnetism and of magnetic strength of
field.   A homogeneously magnetized body must behave as
if the magnetism were situated only on the surface.

§ 61. By vector-analytical methods, an equation express-
ing Biot and Savart's Law and Ampère's " swimmer's rule "
may be derived for the magnetic strength of field produced
at any arbitrary field-point by an element of a closed electric
current.   This equation further leads to the relations

$$\operatorname{div} \mathbf{H} = 0, \qquad \operatorname{rot} \mathbf{H} = \frac{4\pi}{c}\,\mathbf{i}.$$

The latter equation may also be stated in an integral form,
according to which the work done when a unit magnetic pole
travels once round a current is equal to the current, electro-
magnetically measured, multiplied by 4π.

§ 62. The potential energy of a current in a magnetic field is found to be equal and opposite to the product of the current (in electromagnetic measure) and the flux of magnetic force through the surface enclosed by the current. Similarly directed current-elements must attract each other, whereas oppositely directed ones repel each other, the force in either case being directly proportional to the product of the currents, and inversely to the square of the distance between them. For the density of force in a magnetic field we find the relation

$$\mathbf{q} = \frac{1}{c} \, [\mathbf{iH}],$$

where $\mathbf{i}$ denotes the current density.

§ 63. The Principle of the Conservation of Energy leads to the result that, in the case of an induced current, the induced electromotive force must be equal to the time-rate of change of the magnetic flux of force, apart from a proportionality factor of $1/c$. Moreover, the sense of the induced current must be such that the magnetic force arising from it tends to check the inducing process.

§ 64. In virtue of Ohm's Law (which states that the electromotive force is equal to the product of current and resistance), we may vectorially equate the current density in a conductor to the product of the electric strength of field and the specific conductivity. The work done by a current in a unit of time is found to be equal to the product of the resistance and the square of the current.

§ 65. Every variation in the current in a conductor gives rise, through self-induction, to an electromotive force of magnitude $- L \, . \, dI/dt$, where

$$L = \frac{1}{c^2} \iint \frac{d\mathbf{s} \, d\mathbf{s}'}{r} \, ;$$

L is known as the self-inductance. By combining the law of self-induction with Ohm's Law, we obtain the laws governing the extra current, alternating currents, and condenser oscillations.

CHAPTER IX

Maxwell's Theory of the Electromagnetic Field

§ 66. According to Maxwell's theory, every variation in the electric field-strength in an electromagnetic field is accompanied by an electric displacement current, which possesses the same properties as a conduction current, and

which, either by itself or in conjunction with a conduction current, represents a closed circuit. The density of the displacement current is given by the relation

$$\mathbf{g} = \frac{1}{4\pi}\frac{\partial \mathbf{E}}{\partial t}.$$

§ 67. In free space, Maxwell's equations have the form

$$\operatorname{div}\mathbf{E} = 4\pi\varrho, \qquad \operatorname{div}\mathbf{H} = 0,$$

$$\operatorname{rot}\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t}, \qquad \operatorname{rot}\mathbf{H} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}.$$

§ 68. At every point in an electromagnetic field, there is a flow of energy perpendicular to the directions of the electric and the magnetic strengths of field ; the density of this flow is determined by Poynting's vector

$$\mathbf{S} = \frac{c}{4\pi}[\mathbf{EH}].$$

§ 69. It follows from Maxwell's equations that a propagation of the electric and the magnetic field-strengths in plane waves is possible ; the velocity of these waves in free space must be equal to the electromagnetic constant. In a region which contains no charges, these waves must be purely transverse. Every electric wave is also associated with a magnetic wave of the same period and with the same direction of propagation. The vectors of the electric and the magnetic strengths of field are perpendicular to each other in a plane electromagnetic wave. The electromagnetic pressure of radiation is equal to the energy density.

§ 70. Every moving electric charge represents a convection current, and therefore experiences in an electromagnetic field a mechanical force whose value is

$$\mathbf{K} = Q\mathbf{E} + \frac{Q}{c}[\mathbf{vH}],$$

where Q is the charge, and $\mathbf{v}$ its velocity.

§ 71. In a purely electric field, a charged body describes a parabola (when other forces are absent) ; in a transverse magnetic field, it describes a circular path in a plane normal to the magnetic lines of force.

§ 72. Since every electric charge in motion produces a magnetic field possessing energy, it follows that an expenditure of energy is also necessary to set a charged body in motion. From this we deduce the existence of an apparent electromagnetic mass proportional to the square of the

charge. The value of the electromagnetic mass of a charged spherical surface of radius a is found to be

$$m = \frac{2}{3}\frac{Q^2}{ac^2}.$$

§ 73. The influence of the intervening medium in an electrostatic field manifests itself in a reduction of the electric field-strength in the ratio $1 : \epsilon$, where ϵ is defined as the dielectric constant. The tangential component of the strength of field is continuous in crossing the interface between two media, whereas the normal component undergoes a sudden change. This leads to a refraction of the direction of the electric force.

§ 74. To the dielectric constant corresponds in a magnetic field the magnetic permeability (μ), which expresses the influence of the intervening medium. We must then ascribe a density of moment $\mu I/c$ to the shell which is equivalent to a closed current.

§ 75. In a medium whose constants are ϵ and μ, electromagnetic waves are propagated with a velocity $c/\sqrt{\epsilon\mu}$.

CHAPTER X

Theory of Light

§ 76. The strongest support for the electromagnetic theory of light is to be found, on the one hand, in the experimental agreement between the values of the velocity of light and the electromagnetic constant, and on the other hand, in Hertz's researches. The electromagnetic conception of light waves leads at once to their transverse nature, which had been inexplicable from the point of view of the earlier elastic theory of light.

§ 77. Light possessing a definite state of vibration is said to be polarized. Natural light is also to be regarded as polarized, although no definite state of vibration can be recognized in it, owing to the immensely rapid variations in this state.

§ 78. It is found that the intensity of light is determined by the square of the amplitude of the electric (or of the magnetic) strength of field. The intensity of light varies inversely as the square of the distance from the luminous source. In free space, the mean energy density of a light wave is equal to the square of the electric or the magnetic

amplitude divided by 8π. The pressure of light upon a body which completely absorbs it is equal in magnitude to the energy density.

§ 79. From the superposition both of electric and of magnetic fields follows the optical principle of superposition and, in consequence, the explanation of the interference of light. The phenomenon of Newton's rings enabled Young to calculate the wave-lengths of the various colours. The frequency of visible light thus determined is found to extend from about 400 to 800 billion a second.

§ 80. In the electromagnetic theory of light, the fact that the tangential component of the electric strength of field is continuous in crossing the interface between two insulators suffices, without having recourse to any special hypotheses, for the derivation of the laws of reflection and refraction. It also leads to Maxwell's relation, which states that the dielectric constant of a medium is equal to the square of its refractive index.

§ 81. Fresnel's equations can be derived from the fact that the tangential component of the magnetic field-strength, like that of the electric field-strength, must be continuous in crossing the interface between two media. These equations determine the vectors of the electric amplitude in the reflected and the refracted ray, and hence the intensity and direction of vibration of these two rays.

§ 82. A discussion of Fresnel's equations leads to the explanation of polarization by reflection, and of partial polarization by refraction. The phenomena of the polarization produced by reflection prove that the electric vibrations occur at right angles to the plane of polarization, whereas the magnetic vibrations take place in that plane. On the other hand, Wiener's observations on stationary light waves, which he discovered, show that it is the electric vibrations which produce the photographic and, most probably, also the physiological effects of light.

§ 83. Fresnel's equations may also be applied to the case of total reflection. They then lead to the conclusion that the amplitudes of the components of the reflected electric wave are complex. This means that the components of the reflected ray appear displaced in phase relatively to those of the incident ray, and hence that, in general, totally reflected light is elliptically polarized.

§ 84. Metals, and indeed all substances which absorb light, appear in the electromagnetic theory of light as conductors of electricity. Theory leads to the result that metals

may be regarded as media with complex refractive indices $n' = \alpha - i\beta$. β is the absorption coefficient, and c/α the velocity of light in the metal. According to Drude's equations, $\alpha = \beta = \sqrt{\mu\sigma\tau}$ (where σ is the conductivity) with great approximation. Hence in a metal, the velocity of light and also the absorption depend on the colour. Light reflected from metals is, in general, elliptically polarized. The reflective power of metals, which is always approximately unity, is theoretically given by the relation

$$\varrho = 1 - \frac{2}{\sqrt{\mu\sigma\tau}},$$

which has been experimentally confirmed for long waves.

§ 85. In order to explain the transmission of light through crystals, we have to regard the dielectric constant as a tensor. The velocity with respect to a definite wave-normal is then given by the so-called Fresnel's Law. In general, every wave-normal is associated with two waves polarized at right angles to each other. The application of the general formulæ to uniaxial crystals leads to a distinction between the ordinary and extraordinary waves.

(The magnitudes of vectors are, on principle, denoted by italic letters corresponding to the letters in Clarendon type, which represent the vectors themselves.)

PART I

MECHANICS

Together with the General Theory of Vector Fields, of Vibrations, and of Potential

A, B, C Arbitrary vectors.
F Strength of field.
G Momentum.
K Force.
M Moment of force, or moment of a source-couple.
P Curve potential.
R Resultant force.
U Angular momentum.
V Vector quantity.

a Directed interval.
a Unit-vector.
b Acceleration.
f Moment of velocity.
g Acceleration due to gravity.
i, j, k Fundamental vectors.
n Normal unit-vector.
q Density of force.
r Radius vector.
d**s** Element of path or curve.
u Initial velocity.
v Velocity.
w Angular velocity.

I_{xx} Moment of inertia.
L Kinetic energy
S Scalar.
T Period of revolution.
V Mechanical potential.
W Energy.

X, Y, Z Components of force.
df Element of surface.
g Strength of source.
k Constant of damping.
k_{xx} Tensor.
l Length.
m Mass.
p_{xx} Stress.
p Fluid pressure.
q Cross-sectional area.
q Generalized coordinate.
t Time.
t_{xx} Tensor.
x, y, z Coordinates.
Θ Volume dilatation.
Ψ Potential.

α, β, γ Directional cosines.
ϵ Phase constant.
η Energy density.
κ Gravitational constant.
κ Wave velocity.
λ Undetermined multiplier.
λ Wave-length.
λ, μ Elasticity constants.
ϱ Density.
σ Surface density.
$d\tau$ Element of volume.
τ Period.
χ Density of moment.
ω Solid angle.

[1] It was, unfortunately, impossible to avoid the use, in exceptional cases, of certain symbols in another sense than that given in this summary.

Part II

THEORY OF THE ELECTROMAGNETIC FIELD AND OF LIGHT

A Amplitude.
E Electric strength of field.
H Magnetic strength of field.
K Force.
M Magnetic moment.
P Curve potential.
S Poynting's vector.

———

b Acceleration.
f Unit-vector in the direction of propagation.
h Specific magnetization.
i Current density.
n Normal unit-vector.
q Density of force.
r Radius-vector.
d**s** Element of a conductor.
v Velocity.

———

C Capacity.
I Current strength.
K Electromotive force.
L Self-inductance.
Q Charge.
R Resistance.
V Potential energy.

———

c Electromagnetic constant, and velocity of light.
e Electric charge.
df Surface-element.
m Mass.
m Quantity of magnetism.
n Refractive index.
p_{xx} Stress.
q Cross-sectional area.
t Time.

———

Ψ Potential.

———

β Absorption coefficient.
γ Specific charge.
δ Difference in phase.
ϵ Dielectric constant.
η Energy density.
λ Wave-length.
λ Specific conductivity.
μ Magnetic permeability.
ν Frequency.
ϱ Density of charge.
σ Surface density.
σ Specific conductivity.
τ Period.
$d\tau$ Element of volume.
χ Density of moment.

NAME INDEX

327

SUBJECT INDEX

	Date Due		
FEB 2 1960			
AG 11 '60			
NOV 6 1981			

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